Journal of Computational Mathematics
Vol.34, No.4, 2016, 437–450.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1601-m2015-0416

A MODIFIED HSS ITERATION METHOD FOR SOLVING THE COMPLEX LINEAR MATRIX EQUATION $AXB = C$ *

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Abstract

In this paper, a modified Hermitian and skew-Hermitian splitting (MHSS) iteration method for solving the complex linear matrix equation $AXB = C$ has been presented. As the theoretical analysis shows, the MHSS iteration method will converge under certain conditions. Each iteration in this method requires the solution of four linear matrix equations with real symmetric positive definite coefficient matrices, although the original coefficient matrices are complex and non-Hermitian. In addition, the optimal parameter of the new iteration method is proposed. Numerical results show that MHSS iteration method is efficient and robust.

Mathematics subject classification: 65F10, 65F50.
Key words: MHSS iteration method, HSS iteration method, Linear matrix equation.

1. Introduction

In this paper, we consider the following linear matrix equation:

$$AXB = C, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{n\times n}$ are given matrices. Assume that A, B and C are large and sparse matrices, and let $A = W + \mathrm{i}T$, $B = U + \mathrm{i}V$, where $W, T, U, V \in \mathbb{R}^{n\times n}$ are real symmetric matrices, and T, V are positive definite, W, U are positive semidefinite. Then the matrix A is non-Hermitian. As a special case of the coupled Sylvester equations

$$\sum_{j=1}^{n} A_{ij} X_j B_{ij} = C_i \quad (i = 1, \cdots, m),$$

the complex symmetric linear matrix equation (1.1) arises in many problems of scientific computation and engineering applications. Its exact solution problems and the least-squares problems have been discussed in the areas of stability of linear systems [28,29], power systems [35], linear algebra [30], FFT-based solution of certain time-dependent PDEs [22]. Generally speaking,

* Received October 8, 2015 / Revised version received January 4, 2016 / Accepted January 29, 2016 / Published online June 28, 2016 /

the sizes of A and B are usually very large and how to effectively solve this kind of equations involving literally hundreds or thousands of variables is under research.

As well known, the complex linear matrix equation (1.1) is mathematically equivalent to the following linear systems of equations

$$\mathcal{A}x = f, \tag{1.2}$$

where $\mathcal{A} = B^T \otimes A$, and the vectors x and f contain the concatenated columns of the matrices X and C, respectively, with $\otimes$ being the Kronecker product and B^T representing the transpose of the matrix B [21]. However, it is a numerically poor way to determine the solution X of the complex linear matrix equation (1.1), as numerically solving the linear system of equations (1.2) is quite costly and ill-conditioned.

With the application of Kronecker products, some algorithms have been proposed to compute the solution of the linear matrix equation (1.1), see, e.g., [4,5,23,24,37,41,48,49]. Moreover, some efficient methods have been presented to solve the linear and nonlinear matrix equations [36, 44, 45]. The HSS iteration method for non-Hermitian positive definite linear systems of equations was firstly proposed by Bai, et al. in [3], and then it was extended to other equations and conditions. We refer to [7–14, 16–20, 33, 34, 43] and the references therein. However, using the idea of HSS iteration method to solve matrix equation has not been investigated except for the work in [2,40,42,46,50,51]. In [6], Bai, et al. presented a modified HSS iteration method for complex symmetric linear systems of equations. In this paper, we used the similar idea for solving complex symmetric linear matrix equation $AXB = C$ and presented a modified Hermitian and skew-Hermitian splitting (MHSS) iteration method. The linear matrix equation $AXB = C$ is solved iteratively without using the Kronecker product, but adopt a new inner-outer iteration strategy. Although we use an inner-outer iteration strategy, the new HSS iteration method still preserves the convergence property of the "one-level" HSS iteration method without showing the effect of the inner iteration. In the MHSS iteration method, only two linear sub-systems with real and symmetric positive definite coefficient matrices need to be solved at each step instead of the solution of the shifted skew-Hermitian sub-systems of the linear matrix equations with coefficient matrices $\alpha I + \mathrm{i}T$ and $\beta I + \mathrm{i}V$. Besides, the computation of $X^{(k+\frac{1}{2})}$ only needs real arithmetic, then the computation of the iterates $X^{(k+1)}$ requires a modest amount of complex arithmetic due to the fact that the right hand side in the corresponding system is complex. Both can be efficiently computed either exactly by a sparse Cholesky factorization or inexactly by a preconditioned conjugate gradient scheme.

Moreover, as the theoretical shows, the MHSS iteration method will converge to the unique solution of the linear matrix equation (1.1) under certain conditions. Theoretical analysis also shows that an upper bound on the contraction factor of the MHSS iteration method depends on the spectra of the Hermitian parts W and U, but is independent on the the spectra of the matrices T, V, A and B, or on the eigenvectors of the matrices W, U, V, T, A and B. We can also give the optimal parameters which minimize the upper bound of the contraction factor.

In the remainder of this paper, a matrix sequence $\{Y^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{n\times n}$ is said to be convergent to a matrix $Y \in \mathbb{C}^{n\times n}$, if the corresponding vector sequence $\{y^{(k)}\}_{k=0}^{\infty} \subseteq \mathbb{C}^{n^2}$ is convergent to the corresponding vector $y \in \mathbb{C}^{n^2}$, where the vectors $y^{(k)}$ and y contain the concatenated columns of the matrices $Y^{(k)}$ and Y, respectively. If $\{Y^{(k)}\}_{k=0}^{\infty}$ is convergent, then its convergence factor and convergence rate are defined as those of $\{y^{(k)}\}_{k=0}^{\infty}$, correspondingly. In addition, we use $\lambda(W)$, $\|W\|_2$ and $\|W\|_F$ to denote the spectrum, the spectral norm, and the Frobenius norm of the matrix $W \in \mathbb{C}^{n\times n}$, respectively. Note that $\| \bullet \|_2$ is also used to represent the 2-norm of