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A FRACTIONAL STOKES EQUATION AND ITS SPECTRAL APPROXIMATION

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Abstract. In this paper, we study the well-posedness of a fractional Stokes equation and its numerical solution. We first establish the well-posedness of the weak problem by suitably define the fractional Laplacian operator and associated functional spaces. The existence and uniqueness of the weak solution is proved by using the classical saddle-point theory. Then, based on the proposed variational framework, we construct an efficient spectral method for numerical approximations of the weak solution. The main contribution of this work are threefold: 1) a theoretical framework for the variational solutions of the fractional Stokes equation; 2) an efficient spectral method for solving the weak problem, together with a detailed numerical analysis providing useful error estimates for the approximative solution; 3) a fast implementation technique for the proposed method and investigation of the discrete system. Finally, some numerical experiments are carried out to confirm the theoretical results.

Key words. Fractional derivative, Stokes equations, well-posedness, spectral method.

1. Introduction

Fractional partial differential equations(FPDEs) are generalizations of the integer-order models, based on fractional calculus. As a useful tool in modelling the phenomenon related to nonlocality and memory effect, the FPDEs have been attracting increasing attention in recent years. They are now finding many applications in a broad range of fields such as control theory, biology, electrochemical processes, viscoelastic materials, polymer, finance, and etc; see, e.g.,[2, 3, 4, 5, 6, 16, 19, 21, 22, 31, 32, 33] and the references therein. In particular, fractional diffusion equations have been frequently used to describe the so-called anomalous diffusion phenomenon; see, e.g., [12, 15, 17, 35, 37, 38].

In this paper, we will consider sub-diffusion problems of the incompressible flows. This problem is related to the Navier-Stokes equation with sub-dissipation, which has been the subject of many research studies in the community of PDE theory. For example, Katz and Pavlović [20] considered the equations $\frac{\partial u}{\partial t} + (-\Delta)^\alpha u + u \cdot \nabla u + \nabla p = 0$, $\nabla \cdot u = 0$ with the initial condition $u(0,x) = u_0(x) \in C_c^\infty(\mathbb{R}^3)$, and proved that for this equation the Hausdorff dimension of the singular set at time of first blow up is at most $5 - 4\alpha$ and the solution has global regularity in the critical and subcritical hyper-dissipation regimes $\alpha \geq 5/4$. Tao [40] considered the global Cauchy problem for the same Navier-Stokes equations in $\mathbb{R}^d, d \geq 3$, and improved the above-mentioned result in the hyper-dissipation regimes under a slightly weaker condition. Yu & Zhai [43] investigated the well-posedness of the fractional Navier-Stokes equations in some supercritical Besov spaces and largest critical spaces. Xiao et al. [41] proved a general global well-posedness result for the fractional Navier-Stokes equations in some critical Fourier-Besov spaces.

On the other side, a considerable body of literature has been devoted to studying numerical methods for the FPDEs, it is impossible to give even a very brief

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review here. Nevertheless, we refer to [26] for a review on the recent progress of high order numerical methods, particularly spectral methods, for the fractional differential equations. The main focus of the current paper is to set up a functional framework for the fractional Stokes equation in bounded domains, and propose a spectral method for its numerical solutions. As it has been well known for the traditional Stokes equation, a suitable variational formulation is essential for spectral methods to be efficient. The suitable weak form relies on the choice of the space pair for the velocity and pressure. The main contribution of this paper includes: first, we introduce the velocity and pressure spaces such that the associated saddle point problem is well-posed; secondly, we construct an efficient spectral method for numerical approximations of the weak solution. Based on the weak formulation and the polynomial approximation results in the related Sobolev spaces, we are able to derive some error estimates. Finally we present an implementation technique of the algorithm, and some numerical results to confirm the theoretical statements.

The rest of the paper is organized as follows: In the next section we recall some notations of fractional calculus and list some lemmas which will be used in the following sections. In Section 3 the fractional Stokes problem is studied and the well-posedness result is established. Then we propose and analyze in Section 4 a stable spectral method based on weak formulation, and derive the error estimates for the numerical solution. In Section 5, we give some implementation details and present the numerical results to support the theoretical statements. In Section 6, we present an extension to the fractional Navier-Stokes equations. Some concluding remarks are given in the final section.

2. Preliminaries

In this section, we present some notations and basic properties of fractional calculus [1, 13, 34, 24, 36]. which will be used throughout the paper. Let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and real numbers respectively, and set $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Let c be a generic positive constant independent of any functions and of any discretization parameters. We use the expression $A \lesssim B$ (respectively, $A \gtrsim B$) to mean that $A \leq cB$ (respectively, $A \geq cB$), and use the expression $A \cong B$ to mean that $A \lesssim B \lesssim A$. In all that follows, without lose of generality, we set $\Lambda = (-1, 1)$ and $\Omega_d = \Lambda^d$. The generic points of Ω_d is denoted by $\boldsymbol{x} = (x_1, \cdots, x_d)$. In some specific occurrences, we may use (x, y) to represent the generic points of Ω_d when $d = 2$ ((x, y, z) when $d = 3$), and use Ω instead of Ω_d to represent the domain for simplification.

Definition 2.1 (RL fractional integral)**.** *Let $f(x)$ be Riemann integrable on (a, b), $-\infty < a < b < \infty$. The left-sided and right-sided Riemann-Liouville fractional integral of order $s > 0$ are defined by*

$$
{}_aI_x^s f(x) := \frac{1}{\Gamma(s)} \int_a^x (x-t)^{s-1} f(t)\, dt, \qquad {}_xI_b^s f(x) := \frac{1}{\Gamma(s)} \int_x^b (t-x)^{s-1} f(t)\, dt
$$

respectively, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 (RL fractional derivative)**.** *For a given $f(x)$, the Riemann-Liouville fractional derivatives ${}_aD_x^s f$ and ${}_xD_b^s f$ of order $s > 0$ are defined by*

$$
{}_aD_x^s f(x) := \frac{d^n}{dx^n} {}_aI_x^{n-s} f(x) = \frac{1}{\Gamma(n-s)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-s-1} f(t)\, dt, \;\; n = \lceil s \rceil,
$$