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Integrable Properties of a Variant of the Discrete Hungry Toda Equations and Their Relationship to Eigenpairs of Band Matrices

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Abstract. The Toda equation and its variants are studied in the filed of integrable systems. One particularly generalized time discretisation of the Toda equation is known as the discrete hungry Toda (dhToda) equation, which has two main variants referred to as the $\mathrm{dhToda_I}$ equation and $\mathrm{dhToda_{II}}$ equation. The dhToda equations have both been shown to be applicable to the computation of eigenvalues of totally nonnegative (TN) matrices, which are matrices without negative minors. The $\mathrm{dhToda_I}$ equation has been investigated with respect to the properties of integrable systems, but the $\mathrm{dhToda_{II}}$ equation has not. Explicit solutions using determinants and matrix representations called Lax pairs are often considered as symbolic properties of discrete integrable systems. In this paper, we clarify the determinant solution and Lax pair of the $\mathrm{dhToda_{II}}$ equation by focusing on an infinite sequence. We show that the resulting determinant solution firmly covers the general solution to the $\mathrm{dhToda_{II}}$ equation, and provide an asymptotic analysis of the general solution as discrete-time variable goes to infinity.

AMS subject classifications: 15A15, 15A18, 37K10, 37K40, 39A12, 40A05, 45M05, 65F15

Key words: Discrete hungry Toda equation, determinant solution, Lax pair, asymptotic behavior.

1. Introduction

The discrete Toda (dToda) equation is a well-known symbolic discrete integrable system [7], and is also simply the recursion formula of the famous quotient-difference (qd)

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algorithm for computing eigenvalues of symmetric positive-definite tri-diagonal matrices [10]. The dToda equation is deeply related to the discrete Lotka-Volterra (dLV) system, which represents simple predator-prey interactions [12]. When species in the system prey on multiple other species, the dLV is extended to the discrete hungry Lotka-Volterra (dhLV) system [1, 8]. Similarly to the dToda equation, the discrete hungry Toda (dhToda) equations, which are regarded as generalizations of the dToda equation, are closely related to the dhLV systems. One of the dhToda equations closely correlates to the dhLV system in which only a comparatively small number of species survive as discrete-time goes to infinity [2]. The other corresponds to another dhLV system that has the reverse biological feature, i.e., many species survive as discrete-time goes to infinity [5]. To distinguish between these two dhToda variants, we will refer to them as the dhToda_{I} and $\text{dhToda}_{\text{II}}$ equations, respectively. The dhToda_{I} and $\text{dhToda}_{\text{II}}$ equations are respectively written with the involvement of an arbitrary integer M as

$$\begin{cases} q_k^{(n+M)} + e_{k-1}^{(n+1)} = q_k^{(n)} + e_k^{(n)}, \quad k = 1, 2, \cdots, m, \quad n = 0, 1, \cdots, \\ e_k^{(n+1)} q_k^{(n+M)} = q_{k+1}^{(n)} e_k^{(n)}, \quad k = 1, 2, \cdots, m-1, \quad n = 0, 1, \cdots, \\ e_0^{(n)} = 0, \quad e_m^{(n)} = 0, \quad n = 0, 1, \cdots, \end{cases} \tag{1.1}$$

and

$$\begin{cases} q_k^{(n+1)} + e_{k-1}^{(n+M)} = q_k^{(n)} + e_k^{(n)}, \quad k = 1, 2, \cdots, m, \quad n = 0, 1, \cdots, \\ e_k^{(n+M)} q_k^{(n+1)} = q_{k+1}^{(n)} e_k^{(n)}, \quad k = 1, 2, \cdots, m-1, \quad n = 0, 1, \cdots, \\ e_0^{(n)} = 0, \quad e_m^{(n)} = 0, \quad n = 0, 1, \cdots, \end{cases} \tag{1.2}$$

where the subscripts and superscripts with parentheses denote discrete-spatial and discrete-time variables, respectively. Both the dhToda equations with $M = 1$ become the simple dToda equation. In fact, dhToda_{I} Eq. (1.1) was originally derived from a study of a box-ball system, and some of the integrable properties have already been found without consideration of the dhLV systems [13]. However, the $\text{dhToda}_{\text{II}}$ equation has not yet been examined from the viewpoint of discrete integrable systems.

The dhToda equations have both been shown to be applicable to the computation of eigenvalues of totally nonnegative (TN) matrices, where the minors are all positive [3, 11]. It is well-known that symmetric positive-definite tri-diagonal matrices are specializations of TN matrices. Thus, the qd algorithm can be considered to be a specialization of two algorithms designed from the dhToda equations. We hereinafter focus only on the $\text{dhToda}_{\text{II}}$ Eq. (1.2). The $\text{dhToda}_{\text{II}}$ Eq. (1.2) admits a matrix representation called the Lax representation,

$$L^{(n+M)} R^{(n+1)} = R^{(n)} L^{(n)}, \quad n = 0, 1, \cdots, \tag{1.3}$$

where $L^{(n)}$ and $R^{(n)}$ are bi-diagonal matrices given using the $\text{dhToda}_{\text{II}}$ variables $q_k^{(n)}$ and