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Convergence of Recent Multistep Schemes for a Forward-Backward Stochastic Differential Equation

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Abstract. Convergence analysis is presented for recently proposed multistep schemes, when applied to a special type of forward-backward stochastic differential equations (FBSDEs) that arises in finance and stochastic control. The corresponding k-step scheme admits a k-order convergence rate in time, when the exact solution of the forward stochastic differential equation (SDE) is given. Our analysis assumes that the terminal conditions and the FBSDE coefficients are sufficiently regular.

AMS subject classifications: 60H35, 65C20, 60H10

Key words: Convergence analysis, multistep schemes, forward-backward stochastic differential equations.

1. Introduction

Let $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\}_{0\le t\le T})$ be a complete filtered probability space, where $\{\mathscr{F}_t\}_{0\le t\le T}$ represents the filtration generated by a standard d-dimensional Brownian motion defined by the vector $W_t = (W_t^1, W_t^2, \cdots, W_t^d)^T$ in which the superscript T denotes the transpose, and all the P-null sets are augmented to each σ-algebra $\mathscr{F}_t$. We consider the $\{\mathscr{F}_t\}_{0\le t\le T}$-adapted triplet $(X, Y, Z) = \{(X_t, Y_t, Z_t) : t \in [0, T]\}$ that satisfies the following forward backward stochastic differential equations (FBSDEs):

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s & \text{(SDE)}, \\ Y_t = \xi + \int_t^T f(s, X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s & \text{(BSDE)}, \end{cases} \tag{1.1}$$

where T is a fixed positive number, $b : \Omega \times [0, T] \times \mathbb{R}^q \to \mathbb{R}^q$ and $\sigma : \Omega \times [0, T] \times \mathbb{R}^q \to \mathbb{R}^{q\times d}$ are appropriate measurable functions called the drift and diffusion coefficients of the stochastic differential equation (SDE) respectively, $f : \Omega \times [0, T] \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^{p\times d} \to \mathbb{R}^p$

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is some appropriate measurable function called the generator of the backward stochastic differential equation (BSDE), and $\xi \in \mathscr{F}_T$ is its terminal condition.

In 1990, Pardoux & Peng [13] proved the existence and uniqueness of the solutions of the above decoupled FBSDEs. Since then, FBSDEs have been extensively studied, and applications of FBSDEs have been found in many important fields such as mathematical finance, stochastic control, risk measure, partial differential equations (PDEs) and nonlinear expectations and so on [6, 9, 10, 14–16]. There have been many articles on FBSDEs — cf. [8–10, 15] and references therein. In particular, under some standard assumptions Peng [15] proved the nonlinear Feynman-Kac formula

$$Y_s = u(s, X_s), \quad Z_s = \nabla_x u(s, X_s)\sigma(s, X_s), \quad \forall s \in [t, T)\,, \tag{1.2}$$

where $u \in C^{1,2}$ is the solution of the partial differential equation (PDE)

$$L^0_{t,x} u(t,x) + f\big(t, x, u(t,x), \nabla_x u(t,x)\sigma(t,x)\big) = 0 \tag{1.3}$$

with the terminal condition $u(T,x) = \varphi(x)$. Here $L^0_{t,x}$ is the second order differential operator defined by

$$L^0_{t,x} = \frac{\partial}{\partial t} + \sum_i b_i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(t,x)\frac{\partial^2}{\partial x_i \partial x_j}\,. \tag{1.4}$$

These results give a precise link between the solutions of the FBSDEs and partial differential equations. In Ref. [17], some properties of the solutions of FBSDEs under weaker conditions were studied using the Malliavin derivative.

It is well known that analytic solutions (X_t, Y_t, Z_t) of FBSDEs can seldom be expressed in explicit closed-form, so great effort has been devoted to their numerical solution. Some are numerical schemes with low order convergence rates [1, 2, 5, 7, 11, 18] under lower regularity assumptions, while others are high-order numerical methods [4, 19, 21–23]. It is notable that most of the numerical methods are designed for BSDE, or decoupled FBSDEs. It is very difficult to design high-order numerical schemes for coupled FBSDEs as one usually needs to solve the forward SDE with high-order accuracy, which is prevented by the multiple stochastic integral. Recently, a novel high-order multistep numerical scheme for FBSDEs has been proposed [20], where the solution pair (Y, Z) admits high-order convergence rates even when the Euler scheme (a typical low order scheme) is used to solve the SDE.

The main purpose of this work is to prove rigorously the high-order convergence rates of the multistep schemes in Ref. [20]. To demonstrate the main idea, we narrow our discussion to a special type of decoupled FBSDEs which have applications in finance and stochastic contorl, and prove that the k-step scheme admits a k-order convergence rate in time for the solution pair (Y, Z). The analysis is carried out under sufficient regularity assumptions on the terminal conditions and on the coefficients of the FBSDEs. Our analysis invokes numerical ODE theory, a variational formula and Malliavin derivative theory. In Section 2, the multistep schemes introduced in Ref, [20] are briefly reviewed, together with some useful lemmas. Section 3 presents our main result — viz. that the k-step scheme in [20] admits a k-order convergence rate in time. Some concluding remarks are made in Section 4.