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Lie Group Classification for a Generalised Coupled Lane-Emden System in Dimension One

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Abstract. In this article, we discuss the generalised coupled Lane-Emden system $u''+H(v)=0$, $v''+G(u)=0$ that applies to several physical phenomena. The Lie group classification of the underlying system shows that it admits a ten-dimensional equivalence Lie algebra. We also show that the principal Lie algebra in one dimension has several possible extensions, and obtain an exact solution for an interesting particular case via Noether integrals.

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Key words: Lie group classification, generalised Lane-Emden system, principal Lie algebra, equivalence transformations, Lie point symmetries, Noether integrals.

1. Introduction

For various forms of the function $f(y)$ and different values of the integer n, the generalised Lane-Emden equation

$$\frac{d^2y}{dx^2}+\frac{n}{x}\frac{dy}{dx}+f(y)=0 \tag{1.1}$$

has been used to model many phenomena in mathematical physics. When $n=2$ and $f(y)=y^r$ where r is a constant, Eq. (1.1) models the thermal behaviour of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [1–3]. Methods such as Adomian decomposition, numerical, perturbation and homotopy analysis, power series and a variational approach have been used to obtain solutions for this generalised equation — e.g. see [4] and references therein.

Leach [5] studied a modified Emden equation, which led to the symmetry based approach to equations of Lane-Emden-Fowler type. Recently, Lane-Emden systems have been

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used by various researchers to model other physical phenomena, including *inter alia* pattern formation, population evolution and chemical reactions. Existence and uniqueness results have been presented for Lane-Emden systems [7,8], and for other related systems [9–11].

Lie point symmetries of the Lane-Emden system were presented in Ref. [12], where it was shown that a Lie point symmetry of the radial Lane-Emden system

$$\frac{d^2u}{dx^2}+\left(\frac{n-1}{x}\right)\frac{du}{dx}+v^q=0\,,\qquad \frac{d^2v}{dx^2}+\left(\frac{n-1}{x}\right)\frac{dv}{dx}+u^p=0$$

is a Noether symmetry if and only if its parameters belong to the critical hyperbola

$$\frac{n}{p+1}+\frac{n}{q+1}=n-2\,.$$

A complete group classification of the nonlinear Lane-Emden system in dimension one

$$-\frac{d^2u}{dx^2}=v^q\,,\qquad -\frac{d^2v}{dx^2}=u^p \tag{1.2}$$

determined its Lie point symmetries, Noether symmetries and corresponding first integrals [13].

In this article, we discuss a generalisation of the system (1.2) where more general functions replace the power functions v^q and u^p, respectively. Thus we consider the generalised Lane-Emden system

$$\frac{d^2u}{dx^2}+H(v)=0,\qquad \frac{d^2v}{dx^2}+G(u)=0\,, \tag{1.3}$$

where $H(v)$ and $G(u)$ are arbitrary functions of v and u, respectively. Our main objective is to perform Lie group classification of the system (1.3). Group classification was first discussed by Lie [17], and many researchers have since applied Lie's methods to a wide range of problems. The group classification of the system (1.3) involves finding the Lie point symmetries of this system with arbitrary functions $H(v)$ and $G(u)$, and then determining all possible forms of $H(v)$ and $G(u)$ for which the symmetry group can be extended — cf. Ref. [18], and for applications of Lie group analysis to differential equations see Refs. [19–21] for example.

Although the group classification of the generalised Lane-Emden system

$$\frac{d^2u}{dx^2}+\frac{n}{x}\frac{du}{dx}+H(v)=0\,,\qquad \frac{d^2v}{dx^2}+\frac{n}{x}\frac{dv}{dx}+G(u)=0 \tag{1.4}$$

was performed in Ref. [14], and this system (1.3) can formally be viewed as a particular case (by setting $n=0$) of the generalised Lane-Emden system (1.4), the group classification of (1.4) was found with the restriction $n\neq 0$. Our discussion of the system (1.3) here is thus complementary to the previous discussion of the generalised Lane-Emden system (1.4), and it emerges that the principal Lie algebra for (1.3) is one-dimensional whereas it is trivial for (1.4). For completeness, we also present results on the Noether cases, which