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Analysis of $L1$-Galerkin FEMs for Time-Fractional Nonlinear Parabolic Problems

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Abstract. This paper is concerned with numerical solutions of time-fractional nonlinear parabolic problems by a class of $L1$-Galerkin finite element methods. The analysis of $L1$ methods for time-fractional nonlinear problems is limited mainly due to the lack of a fundamental Gronwall type inequality. In this paper, we establish such a fundamental inequality for the $L1$ approximation to the Caputo fractional derivative. In terms of the Gronwall type inequality, we provide optimal error estimates of several fully discrete linearized Galerkin finite element methods for nonlinear problems. The theoretical results are illustrated by applying our proposed methods to the time fractional nonlinear Huxley equation and time fractional Fisher equation.

AMS subject classifications: 65M06, 35B65

Key words: Time-fractional nonlinear parabolic problems, $L1$-Galerkin FEMs, Error estimates, discrete fractional Gronwall type inequality, Linearized schemes.

1 Introduction

In this paper, we study numerical solutions of the time-fractional nonlinear parabolic equation

$$ {}_0^C\mathcal{D}_t^\alpha u - \Delta u = f(u), \quad x \in \Omega \times (0,T] \tag{1.1}$$

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with the initial and boundary conditions, given by

$$\begin{aligned} & u(x,0)=u_0(x), && x\in\Omega, \\ & u(x,t)=0, && x\in\partial\Omega\times[0,T], \end{aligned} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^d$ ($d=1$, 2 or 3) is a bounded and convex polygon/polyhedron. The Caputo fractional derivative ${}_0^C\mathcal{D}_t^\alpha$ is defined as

$${}_0^C\mathcal{D}_t^\alpha u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}\frac{1}{(t-s)^\alpha}ds, \quad 0<\alpha<1. \tag{1.3}$$

Here $\Gamma(\cdot)$ denotes the usual gamma function.

The model (1.1) is used to describe plenty of nature phenomena in physics, biology and chemistry [10,15,24,30]. In the past decades, developing effective numerical methods and rigorous numerical analysis for the time-fractional PDEs have been a hot research spot [6, 8, 11, 16, 25, 28, 33, 35–38]. Numerical methods can be roughly divided into two categories: indirect and direct methods. The former is based on the solution of an integro-differential equation by some proper numerical schemes since time-fractional differential equations can be reformulated into integro-differential equations in general, while the latter is based on a direct (such as piecewise polynomial) approximation to the time-fractional derivative [4, 5, 17, 18].

Direct methods are widely used in practical computations due to its ease of implementation. One of the most commonly used direct methods is the so-called $L1$-scheme, which can be viewed as a piecewise linear approximation to the fractional derivative [29] and which has been widely applied for solving various time-fractional PDEs [9,12]. However, numerical analysis for direct methods is limited, even for a simple linear model (1.1) with

$$f(u)=L_0u, \qquad t\in(0,T]. \tag{1.4}$$

The analysis of $L1$-type methods for the linear model was studied by several authors, while the convergence and error estimates were obtained under the assumption that

$$L_0\leq 0 \tag{1.5}$$

in general, see [13, 14, 23, 31]. The proof there cannot be directly extended to the case of $L_0>0$. Recently, the condition (1.5) was improved in [34], in which a time-fractional nonlinear predator-prey model was studied by an $L1$ finite difference scheme and $f(u)$ was assumed to satisfy a global Lipschitz condition. The stability and convergence were proved under the assumption

$$T^\alpha<\frac{1}{L\Gamma(1-\alpha)}. \tag{1.6}$$