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On Finite Groups Whose Nilpotentisers Are Nilpotent Subgroups

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Abstract. Let G be a finite group and $x \in G$. The nilpotentiser of x in G is defined to be the subset $Nil_G(x) = \{y \in G : \langle x,y \rangle \; is\, nilpotent\}$. G is called an $\mathcal{N}$-group ($\mathfrak{n}$-group) if $Nil_G(x)$ is a subgroup (nilpotent subgroup) of G for all $x \in G \setminus Z^*(G)$ where $Z^*(G)$ is the hypercenter of G. In the present paper, we determine finite $\mathcal{N}$-groups in which the centraliser of each noncentral element is abelian. Also we classify all finite $\mathfrak{n}$-groups.

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1 Introduction

Consider $x \in G$. The centraliser, nilpotentiser and engeliser of x in G are

$$C_G(x) = \{y \in G : \langle x,y \rangle \; is\, abelian\}, \; Nil_G(x) = \{y \in G : \langle x,y \rangle \; is\, nilpotent\}$$

and

$$E_G(x) = \{y \in G : [y,_n x] = 1 \; for\ some\ n\}$$

respectively. Obviously

$$C_G(x) \subseteq Nil_G(x) \subseteq E_G(x) \quad \text{for each } x \in G.$$

Note that $Nil_G(x)$ and $E_G(x)$ are not necessarily subgroups of G. So determining the structure of groups by nilpotentisers (or engelisers) is more complicated than the centralisers. Let G be a finite group. Let $1 \leq Z_1(G) < Z_2(G) < \cdots$ be a series of subgroups of G, where $Z_1(G) = Z(G)$ is the center of G and $Z_{i+1}(G)$, for $i > 1$, is defined by

$$\frac{Z_{i+1}(G)}{Z_i(G)} = Z(\frac{G}{Z_i(G)}).$$

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Let $Z^*(G)=\bigcup_i Z_i(G)$. The subgroup $Z^*(G)$ is called the hypercenter of G. We say a group is $\mathfrak{n}$-group in which $Nil_G(x)$ is a nilpotent subgroup for each $x\in G\backslash Z^*(G)$.

Now a group is $\mathcal{N}$-group in which the nilpotentiser of each element is subgroup and a CA-group is a group in which the centraliser of each noncentral element is abelian (see [16] or [5]). The class of $\mathcal{N}$-groups were defined and investigated by Abdollahi and Zarrin in [1]. In particular they showed that every centerless CA-group is an $\mathcal{N}$-group. In this paper, we shall prove the following generalisation of this result.

Theorem 1.1. *Let G be a nonabelian CA-group. Then G is an $\mathcal{N}$-group if and only if we have one of the following types:*

1. *G has an abelian normal subgroup K of prime index.*
2. $\frac{G}{Z(G)}$ *is a Frobenius group with Frobenius kernel* $\frac{K}{Z}$ *and Frobenius complement* $\frac{L}{Z(G)}$, *where K and L are abelian.*
3. $\frac{G}{Z(G)}$ *is a Frobenius group with Frobenius kernel* $\frac{K}{Z}$ *and Frobenius complement* $\frac{L}{Z(G)}$, *such that* $K=PZ$, *where P is a normal Sylow p-subgroup of G for some prime divisor p of* $|G|$, *P is a CA-group,* $Z(P)=P\bigcap Z$ *and* $L=HZ$, *where H is an abelian* p'*-subgroup of G.*
4. $\frac{G}{Z(G)}\cong PSL(2,q)$ *and* $G'\cong SL(2,q)$ *where* $q>3$ *is a prime-power number and* $16\nmid q^2-1$.
5. $\frac{G}{Z(G)}\cong PGL(2,q)$ *and* $G'\cong SL(2,q)$ *where* $q>3$ *is a prime and* $8\nmid q\pm 3$.
6. $G=P\times A$ *where A is abelian and P is a nonabelian CA-group of prime-power order.*

A group is said to be an E-group whenever engeliser of each element of such group is subgroup. The class of E-groups was defined and investigated by Peng in [13,14]. Also Heineken and Casolo gave many more results about them (see [3,4,6]). Now recall that an engel group is a group in which the engeliser of every elements is the whole group. If G is an E-group such that the engeliser of every element is engel, G is an $\mathfrak{n}$-group since every finite engel group is nilpotent. This result motivates us to classify all finite $\mathfrak{n}$-groups in following theorem.

But before giving it, recall that the Hughes subgroup of a group G is defined to be the subgroup generated by all the elements of G whose orders are not p and denoted by $H_p(G)$ where p is a prime. Also a group G is said to be of Hughes-Thompson type, if for some prime p it is not a p-group and $H_p(G)\neq G$.

Theorem 1.2. *Let G be a nonnilpotent group. Then G is an $\mathfrak{n}$-group if and only if* $\frac{G}{Z^*(G)}$ *satisfies one of the following conditions:*

(1) $\frac{G}{Z^*(G)}$ *is a group of Hughes-Thompson type and*

$$\left|Nil_{\frac{G}{Z^*(G)}}(xZ^*(G))\right|=p$$

for all $xZ^*(G)\in\frac{G}{Z^*(G)}\backslash H_p(\frac{G}{Z^*(G)})$;