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A Goal-Oriented Adaptive Moreau-Yosida Algorithm for Control- and State-Constrained Elliptic Control Problems

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Abstract. In this work, we develop an adaptive algorithm for solving elliptic optimal control problems with simultaneously appearing state and control constraints. The algorithm combines a Moreau-Yosida technique for handling state constraints with a semi-smooth Newton method for solving the optimality systems of the regularized sub-problems. The state and adjoint variables are discretized using continuous piecewise linear finite elements while a variational discretization concept is applied for the control. To perform the adaptive mesh refinements cycle we derive local error estimators which extend the goal-oriented error approach to our setting. The performance of the overall adaptive solver is assessed by numerical examples.

AMS subject classifications: 49J20, 49K20, 65K10, 65N50

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1 Introduction

Optimal control problems with state constraints have been the topic of an increasing number of theoretical and numerical studies. The challenging character of these problems roots in the fact that state constraints feature low regular Lagrange multipliers [4,8]. This low regularity does not allow a pointwise interpretation which complicates not only the analysis of these problems but also their numerical treatment as well. In addition, in the presence of control constraints, the solution may exhibit subsets of the underlying domain where both control and state are active simultaneously. In this case, the uniquen-

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ess of Lagrange multipliers can not be guaranteed [28], which yields to undetermined optimality systems. To overcome these difficulties several techniques in the literature have been proposed. Very popular are relaxation concepts for state constraints such as Lavrentiev, interior point and Moreau-Yosida regularization. The former one is investigated by Tröltzsch in [29] and together with Meyer and Rösch in [23]. Barrier methods in function space (see [33]) applied to state constrained optimal control problems are considered by Schiela in [26]. Relaxation by Moreau-Yosida regularization is matter of subject for the fully discrete case in [3,5] as well as in function space in the work [17] by Hintermüller and Kunisch. Recently in [18] a generalized Moreau-Yosida-based framework also applies for constraints on the gradient of the state. However, as far as we are concerned with adaptive approaches, experiences with this type of problems stay limited. Residual-type a posteriori error estimators for mixed control-state constrained problems are derived in [20]. On the other hand the dual weighted residual method proposed in [1] is applied to derive goal-oriented adaptive meshes for better resolving a certain quantity of interest. This technique is extended for pde-constrained optimization to the presence of control constraints in [13,31] as well as to state constraints in [2,12,14]. Within the framework of goal-oriented adaptive function space algorithms a Lavrentiev regularization approach is considered in [15,20] while an adaptive interior point method is proposed in the works [27,34].

In this work, we design an adaptive finite element algorithm to solve elliptic optimal control problems with control and pointwise state constraints. Following [18], our algorithm combines a Moreau-Yosida regularization approach with a semi-smooth Newton solver [16]. We apply the variational discretization concept [10,19] to the state equation of the regularized optimal control problem. Moreover, for a fixed regularization parameter, we develop a goal-oriented a posteriori error estimate to assess the performance of the variational discretization in terms of the objective functional. We therefore derive a regularized extension of the error representation obtained in [12] to the control and state constrained case. In particular no residual associated to the first order optimality condition with respect to the control appears in our approach. We mention here that we are not interested in the error involved by the regularization parameter. Our aim is rather performing a first attempt to understand the behaviour of a goal-oriented based error estimate in connection with a Moreau-Yosida regularization. An overall error reduction which ties the regularization parameter with the current mesh size is subject of an ongoing research work.

The rest of this paper is organized as follows: In the next section we present the optimal control problem under consideration and recall its first order necessary optimality system. In Section 3 we introduce the regularized version of the problem and state the main convergence theorem. In Section 4 we first apply variational discretization to the regularized sub-problems and propose a semi-smooth Newton solver for the resulting discrete systems. In Section 5 we derive the error representation in terms of the objective functional and we adress the related implementation issues. Finally, numerical examples are reported in Section 6.