Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2017.s10

Vol. **10**, No. 2, pp. 420-436
May 2017

Mesh Quality and More Detailed Error Estimates of Finite Element Method

Yunqing Huang*, Liupeng Wang and Nianyu Yi

Hunan Key Laboratory for Computation and Simulation in Science and Engineering, School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, China

Received 8 November 2016; Accepted (in revised version) 21 January 2017

Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

Abstract. In this paper, we study the role of mesh quality on the accuracy of linear finite element approximation. We derive a more detailed error estimate, which shows explicitly how the shape and size of elements, and symmetry structure of mesh effect on the error of numerical approximation. Two computable parameters G_e and G_v are given to depict the cell geometry property and symmetry structure of the mesh. In compare with the standard a priori error estimates, which only yield information on the asymptotic error behaviour in a global sense, our proposed error estimate considers the effect of local element geometry properties, and is thus more accurate. Under certain conditions, the traditional error estimates and supercovergence results can be derived from the proposed error estimate. Moreover, the estimators G_e and G_v are computable and thus can be used for predicting the variation of errors. Numerical tests are presented to illustrate the performance of the proposed parameters G_e and G_v.

AMS subject classifications: 65N15, 65N30, 65N50

Key words: Finite element methods, error estimation, cell geometry, mesh quality.

1. Introduction

We discuss the role of mesh quality on the accuracy of finite element approximation on triangular meshes. The existing a priori error estimate are in a asymptotic sense

$$\|u - u_h\|_V \le Ch^m,$$

where V is an appropriate space, C and m are some positive real numbers independ of the mesh size h. The asymptotic estimate shows that the difference between an exact

*Corresponding author. *Email addresses:* huangyq@xtu.edu.cn (Y. Q. Huang), wangliupeng@xtu.edu.cn (L. P. Wang), yinianyu@xtu.edu.cn (N. Y. Yi)

solution u and an finite element approximation u_h tends to zero as the mesh size $h \to 0$. In general, it is a very difficult task to explicitly evalute how the constant C depends on the mesh quality. To our best knowledge, the role of the cell gemoetry and mesh quality on the accuracy of approximation is not widely addressed in the literature. Some discussion of this topic can be found in [1, 19]. In [16], Yi and Huang established a fundamental error estimation on general mesh, and designed a computable parameter to depict the role of mesh quality on the accuracy of finite element approximation.

The standard a priori error estimate is often insufficient since: i) it only yield information on the asymptotic error behaviour, and is useless for the adaptive meshes with local refinement; ii) the value of the constant C depends on the exact solution u and can be very large on a mesh with "bad" cell geometry; iii) the value of the constant C also can be infinitesimal on a mesh with higher quality. To make the estimate more accurately, an obvious remedy is to refine the estimation by considering how the C depends on finite element mesh. The question then is how to discribe the mesh quality's effect on the accuracy of finite element approximation and how to quantify the approximation error on a mesh with element size vary greatly. These considerations clearly show the need for an estimation which can show the role of the cell geometry and mesh quality on the accuracy of finite element approximation. Of course, the derived error estimation should be localized and should show explicitly how the constant C depends on the shape and size of elements. Moreover, the calculation of the error estimator can be a priori extracted from the cell geometry and mesh quality.

The objective of this paper is to study the role of cell geometry and the symmetry structure of mesh on the accuracy of finite element method. We discuss how the accuracy of finite element approximation depends on the cell geometry and the mesh structure. In detail, we proposed the following error estimations:

$$\begin{aligned} &\|\nabla(u_I - u_h)\| \lesssim G_e + G_v + h.o.t, \\ &\|u - u_h\|_{\ell^2} \lesssim G_v + h.o.t, \end{aligned}$$

where u_h is the fintie element approximation and u_I is the piecewise linear interpolant of u, h.o.t denotes the higher order terms, and G_e and G_v are defined as follows:

$$G_e = \left(\sum_{e \in \mathcal{E}} |\omega_e| \ell_k^2 \left([\theta_{k+1}]^2 + [\theta_{k-1}]^2 + \left[\frac{\ell_{k+1}}{\ell_k} \right]^2 + \left[\frac{\ell_{k-1}}{\ell_k} \right]^2 \right) \left(1 - \frac{Q_\tau + Q_{\tau'}}{2} \right) \right)^{1/2},$$

$$G_v = \left(\sum_{\tau \in \mathcal{T}_h} |\tau| (\ell_k^2 + \ell_{k-1}^2 + \ell_{k+1}^2)^2 \, (1 - Q_\tau) \right)^{1/2}.$$

The details are stated in Section 2. The error estimations show explicitly how the errors depend on the shape and size of elements, and symmetry of the mesh. The two computable parameters G_e and G_v are proposed for depicting the cell geometry and symmetry structure of the mesh. As a by-product, we are easy to obtain various known superconvergence results based on the presented parameters.