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A Hybrid Spectral Element Method for Fractional Two-Point Boundary Value Problems

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Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

Abstract. We propose a hybrid spectral element method for fractional two-point boundary value problem (FBVPs) involving both Caputo and Riemann-Liouville (RL) fractional derivatives. We first formulate these FBVPs as a second kind Volterra integral equation (VIEs) with weakly singular kernel, following a similar procedure in [16]. We then design a hybrid spectral element method with generalized Jacobi functions and Legendre polynomials as basis functions. The use of generalized Jacobi functions allow us to deal with the usual singularity of solutions at $t=0$. We establish the existence and uniqueness of the numerical solution, and derive a hp-type error estimates under $L^2(I)$-norm for the transformed VIEs. Numerical results are provided to show the effectiveness of the proposed methods.

AMS subject classifications: 26A33, 30E25, 34A08, 65L60, 65L70
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1. Introduction

This paper is concerned with numerical solutions of the following FBVPs:

$$-{}_0^*D_t^{2-\delta}u(t)+b(t)u'(t)+c(t)u(t)=f(t),\quad t\in(0,T), \tag{1.1}$$

with Robin or Dirichlet boundary conditions

$$u(0)-\alpha_0u'(0)=\gamma_0,\qquad u(T)+\alpha_1u'(T)=\gamma_1, \tag{1.2a}$$
$$u(0)=\gamma_0,\qquad u(T)=\gamma_1, \tag{1.2b}$$

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where $\delta \in (0,1)$, and ${}_0^*D_t^{2-\delta}$ refers to either Caputo or RL fractional derivative of order $2-\delta$ (see (2.2) and (2.4), respectively). The constants α_0, α_1, γ_0, γ_1 and the functions $b(t)$, $c(t)$ and $f(t)$ are given. In the case of (1.2a), we assume that $c(t) \geq 0$ and

$$\alpha_0 \geq \frac{1}{1-\delta} \quad \text{and} \quad \alpha_1 \geq 0. \tag{1.3}$$

The conditions $c(t) \geq 0$ and (1.3) guarantee that (1.1) with (1.2a) satisfies a suitable comparison/maximum principle, from which existence and uniqueness of the solution u of (1.1) (see, Theorem 1 of [16]).

The FBVP (1.1) is motivated by the studies on anomalous diffusion processes, which model the steady state of one-dimensional superdiffusion of particle motion when convection is present see [13, 21]. Similar to the classical diffusion case, closed form solutions are usually not available, and one has to resort to numerical methods. Some recent numerical works for (1.1) include finite difference method and piecewise polynomial collocation methods for FBVPs, see [11, 13, 16, 28] and the references therein.

Two main difficulties in solving fractional PDEs such as (1.1) are: (i) fractional derivatives are non-local operators and generally lead to full matrices; and (ii) their solutions are often singular at the endpoint(s) so polynomial based approximations are not efficient.

Since spectral methods are capable of providing exceedingly accurate numerical results with less degrees of freedoms, they have been widely used for numerical approximations of PDEs, see e.g., [4, 10, 12, 24, 25]. In recent years, spectral methods have been proposed for VIEs with smooth/weakly singular kernels. We refer to [7, 8, 18] for the p version of spectral methods and [27, 29] for the hp-version of spectral collocation methods. However, these methods are based on polynomial basis functions which are not particularly suitable for FBVPs whose solutions are generally non-smooth. In some earlier work [1, 5], the authors employed non polynomial methods for weakly singular VIEs. Very recently, Shen et al. [23, 26] proposed one-step and multi-step spectral Galerkin methods using generalized Jacobi functions for weakly singular VIEs.

The main purpose of this paper is to propose and analyze an efficient hybrid spectral element methods for FBVPs. Our approach is inspired by [16] where the authors reformulated Caputo FBVPs (1.1) with Robin boundary conditions to a second kind VIEs with weakly singular kernel, and proposed a numerical scheme based on piecewise polynomial collocation. The main advantage of this approach is that, instead of solving a two-point FBVP which couples all unknowns together, one can now use a time-marching method for VIEs. The main strategies and contributions are highlighted below:

- We extend the approach in [16] to include RL and Caputo FBVPs with other admissible boundary conditions, and propose hybrid spectral element methods with basis functions that can be tuned to match the singularities of the underlying solutions.