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Ill-posedness of Inverse Diffusion Problems by Jacobi's Theta Transform

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Abstract. The subject is the ill-posedness degree of some inverse problems for the transient heat conduction. We focus on three of them: the completion of missing boundary data, the identification of the trajectory of a pointwise source and the recovery of the initial state. In all of these problems, the observations provide over-specified boundary data, commonly called Cauchy boundary conditions. Notice that the third problem is central for the controllability by a boundary control of the temperature. Presumably, they are all severely ill-posed, a relevant indicator on their instabilities, as formalized by G. Wahba. We revisit these issues under a new light and with different mathematical tools to provide detailed and complete proofs for these results. Jacobi Theta functions, complemented with the Jacobi Imaginary Transform, turn out to be a powerful tool to realize our objectives. In particular, based on the Laptev work [Matematicheskie Zametki **16**, 741-750 (1974)], we provide a new information about the observation of the initial data problem. It is actually exponentially ill-posed.

AMS subject classifications: MASC 65N20, 65F22

Key words: Integral operators, regular kernels, Jacobi transform, separated variables approximation.

1 Introduction

In many areas in sciences and engineering, computational methods for the identification of missing boundary data, of pointwise source or of initial states from Cauchy measurements in transient heat transfer seem recurrent (see [1, 2, 5, 8, 22]). They are among few pertinent ways to proceed, if not the only ones. The distinctive property of these inverse problems is their ill-posedness; they suffer from serious instabilities (see [3, 10, 23, 26]). Careless numerical procedures used for the approximation of these unstable problems

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fail most often. We refer to [11] for a general exposition of the possible regularization remedies. The scope here is the ill-posedness degree in the sense of [28] for the reconstruction problems of either the boundary data, the pointwise source or the initial state. In the proofs proposed here, we show how Jacobi Theta functions help to determine how fast the singular values of the underlying operators decreases toward zero, for each of the inverse problems under scrutiny.

The contents of the paper are as follows. Section 2 is a focus on the identification of a missing boundary data, for the diffusion problem. Using Fourier series, we set the inverse problem as a convolution equation; the kernel being an infinite sum of exponentials. Practicing a zoom on this convolution kernel to especially see its shape at the initial time requires a substantial transformation of it. Applying Laplace's transform to the heat equation, solving it explicitly and using the table of the inverse Laplace transform, we derive a different expansion of that kernel, where the time is inverted in a way. This new expression displays the flatness of the convolution kernel at the initial instant. This statement is enough to ensure the severe ill-posedness or the severe instability of the data completion problem. Section 3 introduces the Jacobi Theta functions and enumerates the identities resulting from the Jacobi Imaginary Transform, resulting itself from Poisson's summation formula. Then we revisit the data completion kernel to show that its transformation can be directly deduced, as a particular example, from the 'inverse' formulas on Jacobi's Theta functions. In Section 4, we investigate the non-linear problem of pointwise source reconstruction and illustrate that the corresponding linearized inverse problem is severely ill-posed. We turn in Section 5 to the observation problem, currently studied as the adjoint of the exact control of the temperature by a boundary control (see [30]). The novelty the analysis ends to is the exponential ill-posedness of the boundary controllability problem. There is no clues that this statement has been seen before.

2 Boundary data completion

Let a rod be geometrically represented by I, the segment $(0,\pi)$ of the real axis and $J=(0,T)$ the time interval. We set $Q=I\times J$. The generic point in I is denoted by x and the time variable is t. Assume now be given a boundary condition η in $L^2(0,T)$. Then, we consider the following heat equation

$$\begin{aligned} &\partial_t y - y'' = 0 && \text{in } Q,\\ &y(0,\cdot)=\eta(\cdot), \qquad y'(\pi,\cdot)=0 && \text{on } J,\\ &y(\cdot,0)=0 && \text{on } I. \end{aligned} \tag{2.1}$$

The symbol $'$ is used for the space derivative ∂_x. Putting the source data and the initial state to zero is chosen only for simplicity.

The inverse problem of the boundary completion consists in recovering the data η at extremity $x=0$, which is inaccessible, for some practical reason. Hopefully, it is achieved by collecting observations on y at the other extremity $x=\pi$ where measurements can