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# SATURATION AND RELIABLE HIERARCHICAL A POSTERIORI MORLEY FINITE ELEMENT ERROR CONTROL*

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**Abstract**

This paper proves the saturation assumption for the nonconforming Morley finite element discretization of the biharmonic equation. This asserts that the error of the Morley approximation under uniform refinement is strictly reduced by a contraction factor smaller than one up to explicit higher-order data approximation terms. The refinement has at least to bisect any edge such as red refinement or 3-bisections on any triangle.

This justifies a hierarchical error estimator for the Morley finite element method, which simply compares the discrete solutions of one mesh and its red-refinement. The related adaptive mesh-refining strategy performs optimally in numerical experiments. A remark for Crouzeix-Raviart nonconforming finite element error control is included.

*Mathematics subject classification:* 65M12, 65M60, 65N25.
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## 1. Introduction

The saturation assumption is made in many engineering finite element applications and is often observed in the asymptotic regime for very fine meshes. The mathematical justification is less obvious and often requires restrictions on the mesh-refinement and on extra data oscillations or data approximation terms. Given the two finite element approximations $u_H$ and $u_h$ with respect to a coarse mesh $\mathcal{T}_H$ and its overall refinement $\mathcal{T}_h$ to the exact solution $u$, the errors in the broken energy norm $\| \bullet \|_{\mathrm{NC}}$ (with respect to piecewise Sobolev norms) satisfies

$$\|u - u_h\|_{\mathrm{NC}} \le \varrho\, \|u - u_H\|_{\mathrm{NC}} + C \,\mathrm{data\ apx}(\mathcal{T}_H). \tag{1.1}$$

with positive constants $\varrho < 1$ and $C < \infty$. The data approximation terms data apx$(\mathcal{T}_H)$ read $\|H^\alpha f\|$ for the given right-hand side $f \in L^2(\Omega)$ of the PDE in the $L^2$ norm $\|\bullet\|$ over the domain $\Omega$ weighted by the piecewise constant mesh-size $H$. They can be evaluated explicitly and reflect the mesh-refinement to resolve the local mesh refinement through the variable mesh-size $H$ and

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are of higher order with $\alpha = 2$ for the Morley and with $\alpha = 1$ of first-order for the Crouzeix-Raviart finite element method. Those terms are efficient in the sense data $\mathrm{apx}(\mathcal{T}_H)$ is controlled by the error $\|u-u_h\|_{\mathrm{NC}}$ plus data oscillation terms like $\|H^\alpha(f-\Pi_0 f)\|$ with the piecewise integral means $\Pi_0 f$ of $f$. It is known that $\|H^\alpha(f-\Pi_0 f)\|$ can dominate the error and even $u_H = u_h$ is possible for highly oscillating data $f \in L^2(\Omega)$ in a possibly very large computational regime which makes (1.1) less useful, so this paper aims at applications for piecewise smooth data when this term is negligible. Saturation results of the type (1.1) are justified for the conforming finite element method [5, 12], where counterexamples are characterized for very coarse meshes when (1.1) fails even for a constant right-hand side.

In contrast to [12] for conforming FEMs and second-order problems, this paper asserts saturation for uniform mesh-refinement rather than for an increased polynomial degree. For conforming finite elements for the Poisson equation, (1.1) was recently characterized in [5]. It came as a surprise to the authors that there are no restrictions on the mesh for the nonconforming Morley or Crouzeix-Raviart finite element schemes as all. Moreover, for those schemes, the main result (1.1) of this paper is not restricted to newest-vertex bisection or red-green-blue refinement, but is also valid for more exotic refinement strategies as long as the family $\mathbb{T}$ of triangulations under consideration is shape regular—so unstructured grids with local mesh-refining are included.

An immediate consequence of saturation is hierarchical error control with a justification via a triangle inequality. This and (1.1) imply

$$\|u-u_H\|_{\mathrm{NC}} \le \|u-u_h\|_{\mathrm{NC}} + \|u_H-u_h\|_{\mathrm{NC}} \le \varrho\,\|u-u_H\|_{\mathrm{NC}} + \eta + \mu$$

for the hierarchical error estimator $\eta := \|u_H-u_h\|_{\mathrm{NC}}$ and the data approximation term $\mu := C\,\mathrm{data\ apx}(\mathcal{T}_H)$. Since $\varrho < 1$, this is reliability in the form

$$\|u-u_H\|_{\mathrm{NC}} \le C_{\mathrm{rel}}(\eta+\mu) \quad \text{with reliability constant } C_{\mathrm{rel}} := 1/(1-\varrho). \tag{1.2}$$

The point is that (1.2) is *not* an asymptotic result and holds for all coarse meshes $\mathcal{T}_H$ with the extra cost of calculating $u_h$ with respect to a uniform refinement $\mathcal{T}_h$ thereof. Moreover, the regularity of the exact solution does not enter at all and the higher-order term $\mu$ depends explicitly on the data and can be computed. In conclusion, this paper justifies hierarchical error control in the form

$$\|u-u_H\|_{\mathrm{NC}} \le C_1\|u_h-u_H\|_{\mathrm{NC}} + C_2\mathrm{data\ apx}(\mathcal{T}_H) \tag{1.3}$$

with universal reliability constants $C_1$ and $C_2$. The estimate (1.3) serves as a basis of further more local versions of hierarchical error control with less computational costs as outlined in [29] for conforming finite elements in second-order problems.

The remaining parts of this paper are organized as follows. Section 2 establishes the notation and the main saturation result (1.1) for the biharmonic equation with homogeneous boundary conditions and its numerical simulation with the Morley finite element method. The arguments rely on a new discrete efficiency and a known quasi-orthogonality estimate. Section 3 states the hierarchical error control (1.3) for the Morley finite element method, which is exemplified in numerical experiments in Section 4. Some comments on the second-order Poisson model problem and its numerical simulation with the Crouzeix-Raviart finite element method in Section 5 conclude the paper.

The results are given in two space dimensions for the simplicity of the presentation but are expected to carry over in higher space dimensions.