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# $C^0$ DISCONTINUOUS GALERKIN METHODS FOR A PLATE FRICTIONAL CONTACT PROBLEM*

Fei Wang
*School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China*
*Email: feiwang.xjtu@xjtu.edu.cn*
Tianyi Zhang
*Program in Applied Mathematical and Computational Sciences, University of Iowa, Iowa City, IA 52242, USA*
*Email: tianyi.zhang1106@gmail.com*
Weimin Han
*School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China*
*Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA*
*Email: weimin-han@uiowa.edu*

**Abstract**

Numerous $C^0$ discontinuous Galerkin (DG) schemes for the Kirchhoff plate bending problem are extended to solve a plate frictional contact problem, which is a fourth-order elliptic variational inequality of the second kind. This variational inequality contains a non-differentiable term due to the frictional contact. We prove that these $C^0$ DG methods are consistent and stable, and derive optimal order error estimates for the quadratic element. A numerical example is presented to show the performance of the $C^0$ DG methods; and the numerical convergence orders confirm the theoretical prediction.

*Mathematics subject classification:* 65N30, 49J40
*Key words:* Variational inequality of fourth-order, Discontinuous Galerkin method, Plate frictional contact problem, Optimal order error estimate.

## 1. Introduction

Many problems in physical and engineering sciences are modeled by partial differential equations (PDEs). However, various more complex physical processes are described by variational inequalities (VIs). VIs form an important class of nonlinear problems arising in a wide range of application areas of physical, engineering, financial, and management sciences. In an early reference [6], many problems in mechanics and physics are formulated and studied in the framework of variational inequalities. More detailed study on VIs and numerical methods for solving them can be found in numerous monographs, e.g. [8–11, 14]. In this paper, we study a frictional contact problem for Kirchhoff plates, which is modeled by a fourth-order elliptic variational inequality of the second kind. To solve fourth order elliptic PDEs, the conforming finite element (FE) method uses $C^1$ finite elements, which requires a large number of degrees of freedom and in addition, the method is not easy to implement. To resolve this problem, nonconforming FE methods have been developed, and an early reference on the mathematical analysis of nonconforming FE methods for the plate bending problem is [15]. In [12], nonconforming finite element methods for solving a plate frictional contact problem are analyzed, and

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optimal order error estimates are derived for both continuous and discontinuous nonconforming finite elements. However, the nonconforming FE space needs to be carefully chosen so that the inconsistency error can be controlled. Various discontinuous Galerkin (DG) methods have also been developed to solve fourth order PDEs. DG methods are an important family of nonstandard finite element methods for solving partial differential equations. Discontinuous Galerkin methods use piecewise smooth yet globally less smooth functions to approximate problem solutions, and relate the information between two neighboring elements by numerical traces. The practical interest in DG methods is due to their flexibility in mesh design and adaptivity, in that they allow elements of arbitrary shapes, irregular meshes with hanging nodes, and the discretionary local shape function spaces. In addition, the increase of the locality in discretization enhances the degree of parallelizability. We refer to [5] for a historical account about DG methods.

Due to the inequality form of the problems, the Galerkin orthogonality is lost when DG methods are applied to solve variational inequalities, resulting in substantial difficulty in analyzing DG methods for VIs. Moreover, the bilinear forms of DG schemes are coercive only in the finite element spaces, not in the original function space. Therefore, the standard analytical techniques of finite element methods for VIs are not applicable for DG cases. In [21], numerous DG methods are extended for solving elliptic variational inequalities of 2nd-order, and a priori error estimates are established, which are of optimal order for the linear element. In [22], some discontinuous Galerkin schemes with the linear element for solving the Signorini problem are studied, and an optimal convergence order is proved. The ideas presented in [22] are extended to solve a quasistatic contact problem in [23]. In this paper, we study DG methods to solve a fourth-order elliptic variational inequality of the second kind for the Kirchhoff plates. There are two kinds of DG methods for the biharmonic equations: fully discontinuous ones and $C^0$ continuous ones. The fully discontinuous IP methods are investigated systematically in [16–18, 20] for biharmonic problems. These type DG methods allow meshes with hanging nodes and arbitrary locally varying polynomial degrees on each element, and thus are ideally suited for *hp*-adaptivity, but they still suffer from a large number of degrees of freedom. A $C^0$ IP formulation is introduced for Kirchhoff plates in [7] and quasi-optimal error estimates are obtained for smooth solutions. Unlike fully discontinuous Galerkin methods, $C^0$ type DG methods do not "double" the degrees of freedom on element boundaries. The FE spaces belong to $C^0$, not $C^1$; penalty terms on inter-element boundaries are added to force the derivative to be nearly continuous. Therefore, $C^0$ DG schemes have good accuracy with fewer number of degrees of freedom, leading to the time saving in solving the discretized problems. A rigorous error analysis is presented in [3] for the $C^0$ IP method under weak regularity assumption on the solution. A weakness of this method is that the penalty parameter can not be precisely quantified a priori, and it must be chosen suitably large to guarantee stability. However, a large penalty parameter has a negative impact on accuracy. Based on this observation, another $C^0$ DG method is introduced in [25], where the stability condition can be precisely quantified. In [13], a consistent and stable $C^0$ DG method, called the local $C^0$ DG (LCDG) method, is derived for the Kirchhoff plate bending problem.

In this paper, we consider $C^0$ DG methods to solve the Kirchhoff frictional contact plate problem, which is a fourth-order elliptic variational inequality of the second kind. This model variational inequality arises in the study of a frictional contact problem for Kirchhoff plates. It is difficult to construct stable DG methods for such problems because of the higher order and the inequality form. For fourth-order elliptic variational inequalities of the first kind, some