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NUMERICAL SOLUTIONS OF NONAUTONOMOUS STOCHASTIC DELAY DIFFERENTIAL EQUATIONS BY DISCONTINUOUS GALERKIN METHODS*

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Abstract

This paper considers a class of discontinuous Galerkin method, which is constructed by Wong-Zakai approximation with the orthonormal Fourier basis, for numerically solving nonautonomous Stratonovich stochastic delay differential equations. We prove that the discontinuous Galerkin scheme is strongly convergent, globally stable and analogously asymptotically stable in mean square sense. In addition, this method can be easily extended to solve nonautonomous Stratonovich stochastic pantograph differential equations. Numerical tests indicate that the method has first-order and half-order strong mean square convergence, when the diffusion term is without delay and with delay, respectively.

Mathematics subject classification: 34K50, 60H35, 65L20, 65L60.
Key words: Discontinuous Galerkin method, Wong-Zakai approximation, Nonautonomous Stratonovich stochastic delay differential equation.

1. Introduction

Stochastic delay differential equations (SDDEs) have become an increasingly prominent tool in dynamical systems because they can be applied to modeling the evolution phenomena which contain some noises and delays [1–3]. Although SDDEs can be deemed as a special kind of stochastic differential equations (SDEs), the extension of numerical methods from SDEs to SDDEs is nontrivial since the appearance of time delay influences stability properties, convergence properties, computational complexities and even more [4–6]. To this end, it is of great significance to study alone numerical methods for SDDEs. Recently, for numerically solving SDEs, SDDEs and even neutral SDDEs, many finite difference methods have been extensively studied (e.g., Euler-type schemes [7–9], Milstein-type schemes [10–12], theta-type schemes [13–16], Runge-Kutta schemes [17–21], etc.). The spectral collocation method [22] is also used to solve SDDEs. However, the finite element method is rarely used to SDDEs.

In this paper, we are devoted to developing the discontinuous Galerkin (DG) method (be-

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longing to the finite element method) for a nonautonomous Stratonovich SDDE

$$dX(t) = f(t, X(t), X(t-\tau))dt + \sum_{l=1}^{r} g_l(t, X(t), X(t-\tau)) \circ dW_l(t), \qquad t \in (0, T], \tag{1.1a}$$

$$X(t) = \Psi(t), \qquad t \in [-\tau, 0], \tag{1.1b}$$

where τ is a positive fixed delay, the drift coefficient $f : [0, T] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ and the diffusion coefficient $g_l : [0, T] \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ are Borel measurable. Let

$$(W(t), \mathcal{F}_t) := (\{W_l(t), 1 \le l \le r\}, \mathcal{F}_t)$$

be a system of r-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbf{P})$. Let $\Psi(t)$ be an $\mathcal{F}_0$-measurable $C([-\tau, 0]; \mathbb{R})$-valued random process such that $\mathbb{E}[\|\Psi\|^2] < \infty$, here $C([-\tau, 0]; \mathbb{R})$ is a Banach space of all continuous paths from $[-\tau, 0]$ to $\mathbb{R}$ equipped with the supremum norm.

Unlike traditional continuous Galerkin methods, the DG method works over a trial space of functions that are only piecewise continuous. In 1973, Reed and Hill initially introduced a DG method to solve the hyperbolic neutron transport equation [23]. Nowadays, it has been successfully applied to hyperbolic, elliptic, parabolic and mixed-form problems arising from a wide range of applications. For deterministic delay differential equations [24,25], the DG method is a class of locally conservative, stable, and high-order convergent finite element method. For SDEs with additive noise [26], the DG method is equivalent to an implicit Runge-Kutta method, which is A-stable and convergent of high order. So far, the DG method for SDDEs is not yet considered.

It is well known that Brownian motion can be of crucial importance to SDDE (1.1). Since Brownian motion is nowhere differentiable, it destroys the regularity of solutions of SDDEs such that the DG scheme cannot be applied directly. Motivated by the Wong-Zakai approximation, we can transform SDDEs to delay differential equations with random coefficients, which are of better regularity. In fact, using Wong-Zakai approximation as an intermediate step to design numerical methods is a very natural strategy. Because Brownian motion can only be approximated simulation in any physical experiment and can not be realized in the real sense. In 1965, Wang and Zakai [27,28] established the relationship between ordinary differential equations and SDEs by using piecewise linear interpolation polynomials (which are continuous, of bounded variation, and have a piecewise continuous derivative) to approximate Brownian motion. In 1991, Wong-Zakai approximation has been developed to stochastic functional differential equations [29]. Nowadays, Wong-Zakai approximation has been used to derive some finite difference schemes for SDEs and SDDEs [30,31].

The rest of this paper is organized as follows. In Section 2, Wong-Zakai approximation is introduced and used to design DG methods for nonautonomous SDDE (1.1). Section 3 illustrates that the DG method is strongly mean square convergent. In Section 4, some numerical stability properties are also investigated. Section 5 gives an extension of this method to solve nonautonomous Stratonovich stochastic pantograph differential equations. Numerical experiments will be provided in Section 6 to show convergence order and effectiveness of the finite element method. Finally, Section 7 affords some brief conclusions.