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SUPERCONVERGENCE ANALYSIS FOR TIME-FRACTIONAL DIFFUSION EQUATIONS WITH NONCONFORMING MIXED FINITE ELEMENT METHOD *

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Abstract

In this paper, a fully discrete scheme based on the $L1$ approximation in temporal direction for the fractional derivative of order in $(0,1)$ and nonconforming mixed finite element method (MFEM) in spatial direction is established. First, we prove a novel result of the consistency error estimate with order $O(h^2)$ of EQ_1^{rot} element (see Lemma 2.3). Then, by using the proved character of EQ_1^{rot} element, we present the superconvergent estimates for the original variable u in the broken H^1-norm and the flux $\vec{p}=\nabla u$ in the $(L^2)^2$-norm under a weaker regularity of the exact solution. Finally, numerical results are provided to confirm the theoretical analysis.

Mathematics subject classification: 65N15, 65N30
Key words: Nonconforming MFEM, $L1$ method; Time-fractional diffusion equations, Superconvergence.

1. Introduction

Consider the following time-fractional diffusion equations(TFDEs):

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u - \Delta u = f(X,t), & (X,t)\in\Omega\times(0,T], \\ u(X,t)=0, & (X,t)\in\ \partial\Omega\times(0,T], \\ u(X,0)=u_0(X), & X\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded convex domain with boundary $\partial\Omega$, $X=(x,y), T<\infty$, $u_0(X)$ and $f(X,t)$ are given smooth functions and ${}_0^C\mathcal{D}_t^\alpha$ is Caputo fractional derivative defined by

$${}_0^C\mathcal{D}_t^\alpha u(X,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial u(X,s)}{\partial s}\frac{ds}{(t-s)^\alpha},\qquad 0<\alpha<1.$$

TFDEs, which may describe many physical phenomena in enginering, biology, chemistry, physics and so forth, were derived from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order α $(0<\alpha<1)$ that represents a degree of memory in the diffusing meterial [1]. Numerous effort has been devoted to mathematical study of different fractional equations in two branches in the past several decades.

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One is to find the analytical solutions [1-4] and the other is to seek numerical approximation methods. We refer to [5-16] for finite difference methods, [17-25] for FEMs, [26-29] for spectral methods, [30-32] for DG methods, [33-34] for MFEMs, [35] for adaptive FEM, and so forth. Except for the theoretical analysis, another essential difficulty for time fractional PDEs is the computational cost and storage, especially for the long time simulations of the 2-D and 3-D cases. This requires a fast algorithm to reduce the computational complexity significantly, and keep the almost same accuracy comparing with the direct methods, one can see [36, 37, 38] and references therein.

As we know, superconvergence analysis which aims at improving the accuracy and efficiency of FEMs, has been well studied for TFDEs. For example, [39] analysed a piecewise-linear DG method for a kind of fractional diffusion and wave equations, and proved the superconvergent properties at the element's nodes for the classical heat equation. [40] investigated superconvergence properties of the spectral interpolation involving fractional derivatives. In [41], optimal collocation nodes for fractional derivative operators were examined. In [42], by the $L1$ method in time (also see [24, 25, 27, 43, 44]) combined with quasi-Wilson nonconforming finite element in space for Eq. (1.1), the superclose and superconvergent results in the broken H^1-norm were obtained. As for MFEMs, [45] studied an H^1-Galerkin FEM for Eq.(1.1) in 1-D, and derived the convergent result of $O(\Delta t^{2-\alpha} + h^{r+1}\Delta t^{-\alpha} + h^{r+1})$ in L^2-norm for a fully-discrete scheme. Moreover, [46] developed two fully-discrete schemes for Eq. (1.1) with bilinear finite element and EQ_1^{rot} nonconforming element, respectively. The superclose and superconvergent error estimates of order $O(h^2 + \tau^{2-\alpha})$ for both the original variable u in the H^1-norm and the flux $\vec{p} = \nabla u$ in the L^2-norm were derived for bilinear element. However, only optimal error estimates were deduced for EQ_1^{rot} element. The main reason is that the consistency error of this element is only estimated as

$$\sum_K \int_{\partial K} \frac{\partial u}{\partial \vec{n}} v_h ds \le Ch^2 |u|_3 \|v_h\|_{1,h} \le Ch|u|_3\|v_h\|_0,$$

Here v_h belongs to EQ_1^{rot} finite element space (see [47-49]).

As an attempt, the main aim of this paper is to present the superconvergent estimates for Eq.(1.1) with nonconforming MFEM under a weaker regularity of the solutions. The key to our proof is a novel estimate:

$$\sum_K \int_{\partial K} \frac{\partial u}{\partial \vec{n}} v_h ds \le Ch^2 |u|_4 \|v_h\|_0,$$

which improves the corresponding conclusion obtained in [50], i.e.,

$$\sum_K \int_{\partial K} \frac{\partial u}{\partial \vec{n}} v_h ds \le Ch^2 |u|_5 \|v_h\|_0.$$

Obviously the regularity of $u \in H^5(\Omega)$ is weaken to $u \in H^4(\Omega)$, which improves the results of optimal error estimates in [46] to the superconvergent estimates herein.

The rest of the paper is organized as follows. In Section 2, the nonconforming MFEs and some useful Lemmas are introduced. In Section 3, a fully discrete scheme is established and the superconvergent results are obtained. In the last section, some numerical results are provided to confirm our theoretical analysis.