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The Convexity of a Fully Nonlinear Operator and Its Related Eigenvalue Problem

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Abstract. We first get an existence and uniqueness result for a nonlinear eigenvalue problem. Then, we establish the constant rank theorem for the problem and use it to get a convexity property of the solution.

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Key words: Hessian operator, eigenvalue problem, constant rank theorem, strict convexity.

1 Introduction

Let $\Omega \subset R^3$ be a strictly convex bounded domain in R^3 with smooth boundary. We consider the following eigenvalue problem

$$\begin{cases} \sigma_2(W_{ij}(D^2u)) = \lambda(-u)^2 & in\ \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where $D^2u = (u_{ij})$ is the hessian matrix of u, $(W_{ij}(D^2u))$ is a symmetric matrix defined as

$$(W_{ij}) = \begin{bmatrix} u_{11}+u_{22} & u_{32} & -u_{31} \\ u_{23} & u_{11}+u_{33} & u_{21} \\ -u_{13} & u_{12} & u_{22}+u_{33} \end{bmatrix} \tag{1.2}$$

and σ_2 is the 2-nd hessian operator (i.e. $\sigma_2(S) =$ the sum of the 2-principal minors of S for any 3×3 symmetric matrix S). We first prove an existence and uniqueness result for (1.1). Then we get some convexity result for the solution of it.

The eigenvalue problem played an important role in partial differential equations and had been studied by many authors (see, e.g., [21, 24, 25, 33]). Lions [25] first got

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the existence and uniqueness result for the eigenvalue problem of Monge-Ampère equation. Later on, Wang [33] (and Geng-Yu-Qu [12]) generalized this result to the k-hessian equations. In this paper, we get similar results for (1.1). Another important problem in PDE is the convexity problem, which connects the geometric properties to geometric inequalities. One powerful tool to study the convexity is the constant rank theorem. Caffarelli-Friedman [6] proved a constant rank theorem for convex solutions of quasi-linear elliptic equations in R^2. Meanwhile, a similar result was discovered by Yau [30]. Korevaar-Lewis [23] generalized their results to R^n. Later on, Caffarelli-Guan-Ma [8] and Bian-Guan [4] established the constant rank theorem for a class of fully nonlinear equations. Related to our problems, Liu-Ma-Xu [26] established the constant rank theorem for the eigenvalue problem related to $k-Hessian$ equations for $k=2$ in dimension $n=3$. In this paper, we would get a convexity result for (1.1) similar to [26].

Our another motivation to study (1.1) comes from the concept of $k-convex$ solutions introduced by Harvey-Lawson [18] who introduced some general convexity on the solutions of the nonlinear elliptic Dirichlet problem. In their definition, a C^2 function u is said to be $k-convex$ if the sum of any k eigenvalues of its hessian matrix is nonnegative. Recently, Han-Ma-Wu [16], Tosatti-Weinkove [31] studied a similar "convexity"-the $n-1$ plurisubharmonicity for C^2 functions defined on $\Omega\subset C^n$ (i.e. the sum of any $n-1$ eigenvalues of the complex hessian $(\frac{\partial^2 u}{\partial z_i\partial \bar{z}_j})$ is positive) and [31] used it to study the form-type Calabi-Yau equation (see [10, 11]). The $k-convexity$ is related to (1.1) in the sense that if we note the eigenvalues of D^2u by λ_i, (i=1,2,3). Then, by an orthogonal transformation, it is easy to know that the three eigenvalues of $(W_{ij}(D^2u))$ are $\lambda_1+\lambda_2,\lambda_1+\lambda_3,\lambda_2+\lambda_3$. So, u is $2-convex$ if and only if $(W_{ij}(D^2u))$ is positive semi-definite. For our purpose here, we do not need $(W_{ij}(D^2u))$ to be positive semi-definite. Instead, we only need $(W_{ij}(D^2u))\in\Gamma_2$ (the definition of Γ_2 will be given below) for the operator $F(D^2u)=\sigma_2(W_{ij}(D^2u))$ to be elliptic on u.

We see that the operator $F(D^2u)=\sigma_2(W_{ij}(D^2u))$ is a combination of the $2-hessian$ operator σ_2 and a linear one. In our proof of the theorems, we will use the elementary properties of the hessian operator repeatedly. So, let us state some preliminary knowledge that will be used below.

For $1\le k\le n$, let σ_k be the $k-th$ elementary symmetric function, i.e.

$$\sigma_k(\lambda)=\sum_{1\le i_1<i_2<\cdots<i_k\le n}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k},\qquad \forall\,\lambda=(\lambda_1,\dots,\lambda_n)\in R^n. \tag{1.3}$$

Let S^n be the set of all $n\times n$ real symmetric matrix. For $S\in S^n$, let $\lambda(S)=(\lambda_1,\lambda_2,\dots,\lambda_n)$ be the eigenvalues of S. We use the same notion σ_k to define the $k-hessian$ operator as

$$\sigma_k(S)=\sigma_k(\lambda(S)). \tag{1.4}$$

We denote $\Gamma_k=\{\lambda\in R^n|\sigma_i(\lambda)>0,i=1,\dots,k\}$, which is an open convex cone in R^n. We also denote $\Gamma_k=\{S\in S^n|\lambda(S)\in\Gamma_k\}=\{S\in S^n|\sigma_i(S)>0,i=1,\dots,k\}$ if there is no confusion. It is well known that the $k-hessian$ operator σ_k is elliptic with respect to S in Γ_k and