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On the Triviality of a Certain Kind of Shrinking Solitons

Zhuhong Zhang*

School of Mathematical Sciences, South China Normal University, Guangzhou 510631, P.R. China.

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Abstract. In this paper, we study shrinking gradient Ricci solitons whose Ricci tensor has one eigenvalue of multiplicity at least $n-2$. Firstly, we show that if the minimal eigenvalue of Ricci tensor has multiplicity at least $n-1$ at each point, then the soliton are Einstein. While on the shrinking gradient Ricci solitons whose maximal eigenvalue has multiplicity at least $n-1$, the triviality are also true if we naturally require the positivity of Ricci tensor.
We further prove that if the maximal (or minimal) eigenvalue of Ricci tensor has multiplicity at least $n-2$ at each point , and in addition the sectional curvature is bounded from above, then the soliton are Einstein.

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Key words: Einstein manifold, shrinking gradient Ricci soliton, positive Ricci curvature, pinched sectional curvature.

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1 Introduction

A Riemannian manifold (M,g), coupled with a smooth potential function f, is called a gradient Ricci soliton, if there exists a constant ρ, such that

$$R_{ij}+f_{ij}=\rho g_{ij}.$$

It is called shrinking, steady, or expanding, if $\rho>0$, $\rho=0$, or $\rho<0$, respectively. Note that, when the potential function f is a constant, a gradient Ricci soliton is an Einstein manifold. So they are natural generalizations of Einstein metrics. Gradient Ricci solitons also correspond to self-similar solutions of the Ricci flow, and play a fundamental role in the singularity study of the flow. They have been widely studied (see the survey of Cao [1,2] for nice overviews).

*Corresponding author. *Email address:* `juhoncheung@sina.com` (Z.-H. Zhang)

In particular, compact steady and expanding Ricci solitons are shown to be Einstein by Perelman [10]. For the shrinking case, Ivey [6] proved that any compact three-dimensional shrinking Ricci solitons are Einstein and, up to quotients, isometric to the round sphere S^3. Dimension four is then the lowest dimension allowing "nontrivial" examples of shrinking Ricci solitons and such examples do exist (see the survey [1] and references therein).

So we need more conditions to obtain the triviality of shrinking gradient Ricci solitons. In this decade, many results were built on conditions of Weyl tensor, for instance [5,7,9,11,13]. In particular, a full classification is achieved by author in [12] for locally conformally flat shrinking gradient Ricci solitons. However, since the Ricci flow does not preserve the vanishing of Weyl tensor, thus it is necessary to study solitons with non-vanishing Weyl tensor.

Note that, Petersen and Wylie in [11] found an interesting phenomena, says that a shrinking gradient Ricci soliton whose Ricci tensor has one nonzero eigenvalue (obviously, this eigenvalue is positive) of multiplicity $n-1$ must split. Motivated by this work, we study shrinking gradient Ricci solitons whose Ricci tensor has one eigenvalue of multiplicity at least $n-2$. Our first result is the following.

Theorem 1.1. *Let (M^n,g,f) $(n\geq 3)$ be a complete shrinking gradient Ricci soliton whose maximal eigenvalue of Ricci tensor has multiplicity at least $n-1$ at each point, then the Ricci tensor must be nonnegative. If in addition, assume the Ricci tensor is positive, then the soliton is Einstein.*

Similarly, if the minimal eigenvalue of Ricci tensor has multiplicity at least $(n-1)$, we have a similar result.

Theorem 1.2. *Let (M^n,g,f) $(n\geq 3)$ be a complete shrinking gradient Ricci soliton whose minimal eigenvalue of Ricci tensor has multiplicity at least $n-1$ at each point, then the soliton is Einstein.*

Next, we turn to study shrinking solitons whose maximal eigenvalue of Ricci tensor has multiplicity at least $n-2$. In this situation, we need more curvature assumptions to obtain the triviality of solitons.

Theorem 1.3. *Let (M^n,g,f) $(n\geq 4)$ be a complete shrinking gradient Ricci soliton, and at each point, the maximum eigenvalue of Ricci tensor has multiplicity at least $n-2$. If the sectional curvature satisfies $K\leq \frac{n-2}{n^2(n-3)}R$, then it must be Einstein.*

By a similar argument, we can get a triviality of solitons in the other direction.

Theorem 1.4. *Let (M^n,g,f) $(n\geq 4)$ be a complete shrinking gradient Ricci soliton, and at each point, the minimal eigenvalue of Ricci tensor has multiplicity at least $n-2$. If the sectional curvature satisfies $K\leq \frac{2}{n^2}R$, then it must be Einstein.*