Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **11**, No. 5, pp. 1219-1247

DOI: 10.4208/aamm.OA-2018-0157
October 2019

A New Energy-Preserving Scheme for the Fractional Klein-Gordon-Schrödinger Equations

Yao Shi, Qiang Ma* and Xiaohua Ding*

Department of Mathematics, Harbin Institute of Technology at Weihai, Weihai 264209, Shandong, China

Received 1 July 2018; Accepted (in revised version) 22 February 2019

Abstract. In this paper, we study a fourth-order quasi-compact conservative difference scheme for solving the fractional Klein-Gordon-Schrödinger equations. The scheme constructed in this work can preserve exactly the discrete charge and energy conservation laws under Dirichlet boundary conditions. By the energy method, the proposed quasi-compact conservative difference scheme is proved to be unconditionally stable and convergent with order $\mathcal{O}(\tau^2+h^4)$ in maximum norm. Finally, several numerical examples are given to confirm the theoretical results.

AMS subject classifications: 35R11, 26A33, 35A35, 65M12

Key words: Fractional Klein-Gordon-Schrödinger equations, Riesz fractional derivative, conservative scheme, stability, convergence.

1 Introduction

The coupled Klein-Gordon-Schrödinger (KGS) system is a classical model in quantum mechanics, which describes the Yukawa interaction between the conservative complex nucleon field with the neutral real meson field. The KGS system has been widely applied in many physical fields [1,2]. There has also been a surge in the study of the KGS system (see [3–6] and references therein).

Fractional partial differential equations (FPDEs) have attracted increasing attention due to their numerous applications in many different fields [7–10]. Generally, it is usually difficult to obtain the analytical solutions of FPDEs. Many valuable conclusions have been discovered for solving FPDEs [11–28]. These work is very helpful to our research. Thus there is a need to develop a new numerical scheme for the fractional Klein-Gordon-Schrödinger (FKGS) equations.

*Corresponding author.
Emails: mathshiyao@126.com (Y. Shi), hitmaqiang@hit.edu.cn (Q. Ma), mathdxh@hit.edu.cn (X. H. Ding)

Furthermore, many partial differential equations are known to possess conservative laws that are preserved under suitable analytical conditions, including systems like the Schrödinger, the sine-Gordon and the nonlinear Klein-Gordon equations from relativistic quantum mechanics. As a basic rule, the numerical algorithms should preserve the intrinsic properties of the original problems as much as possible [29]. So, the design of conservative finite difference schemes for partial differential equations is always an important topic in research. Motivated by this fact, for the fractional partial differential equations, it is of interest to investigate conservative finite difference schemes. Therefore, we propose a quasi-compact conservative difference scheme for the FKGS equations and discuss its properties in this paper.

The remainder of this paper is sectioned as follows. The conservative FKGS equations with Riesz space-fractional derivatives is presented in Section 2, which motivates our investigation. The relevant definitions of the fractional differential operators and conservation laws are also shown in this section. A quasi-compact conservative difference scheme is proposed then in Section 3 to approximate solutions of the FKGS equations. In Section 4, we will discuss the preservation of discrete charge and energy conservation laws, stability and convergence of the proposed scheme. Finally, numerical experiments are carried out to confirm our theoretical results and show the efficiency of the proposed scheme.

2 Preliminaries

2.1 The coupled FKGS equations

We start this section with some definitions which are used later.

Definition 2.1 ([30])**.** The left and right Riemann-Liouville fractional derivatives of order α on the infinite interval are be defined as

$$
\begin{aligned}
{}_{-\infty}D_x^{\alpha}u(x,t) &= \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{-\infty}^{x}\frac{u(\xi,t)}{(x-\xi)^{\alpha+1-n}}d\xi,\\
{}_{x}D_{+\infty}^{\alpha}u(x,t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{x}^{+\infty}\frac{u(\xi,t)}{(\xi-x)^{\alpha+1-n}}d\xi,
\end{aligned}
$$

where $\Gamma(\cdot)$ denotes the usual Gamma function and $n-1<\alpha\le n$.

Definition 2.2 ([30])**.** The Riesz fractional derivative of order α on the infinite interval is defined as

$$
\frac{\partial^{\alpha}u}{\partial|x|^{\alpha}}(x,t)=c_{\alpha}\big[{}_{-\infty}D_x^{\alpha}u(x,t)+{}_{x}D_{+\infty}^{\alpha}u(x,t)\big], \tag{2.1}
$$

where $c_{\alpha}=-\frac{1}{2\cos(\frac{\pi\alpha}{2})}$ and $1<\alpha\le 2$.