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A Locking-Free DP-Q2-P1 MFEM for Incompressible Nonlinear Elasticity Problems

Weijie Huang and Zhiping Li*

LMAM & School of Mathematical Sciences, Peking University, Beijing 100871, China

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Abstract. A mixed finite element method (MFEM), using dual-parametric piecewise biquadratic and affine (DP-Q2-P1) finite element approximations for the deformation and the pressure like Lagrange multiplier respectively, is developed and analyzed for the numerical computation of incompressible nonlinear elasticity problems with large deformation gradient, and a damped Newton method is applied to solve the resulted discrete problem. The method is proved to be locking free and stable. The accuracy and efficiency of the method are illustrated by numerical experiments on some typical cavitation problems.

AMS subject classifications: 65N12, 65N30, 74B20, 74G15, 74M99

Key words: DP-Q2-P1 mixed finite element, damped Newton method, locking-free, incompressible nonlinear elasticity, large deformation gradient, cavitation.

1. Introduction

For incompressible elasticity, it is well known that, even in the case of small deformation and linear elasticity, the notorious volume locking can happen and ultimately leads to the failure of some finite element approximations [11–14]. In the case of incompressible linear elasticity, it is well known how to overcome locking numerically, for example, by using the enhanced assumed strain methods to increase the degrees of freedom of the elements [16, 17], by using the nonconforming finite element methods to weakening the global continuity of the numerical solutions [1], or by using the mixed finite element methods (MFEMs) to relax the constraint of the incompressibility on the numerical solutions [11, 18], etc.. However, for incompressible nonlinear elasticity, especially for the large deformation gradient problems which will be addressed in this paper, there still lacks of systematic results.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with smooth boundary occupied by an isotropic hyper-elastic body in its reference configuration. Denote $M_+^{2\times 2}$ as the set of 2×2 matrices with positive eigenvalues, and $\boldsymbol{u}$ as the deformation field. Let the stored

*Corresponding author. *Email address:* `lizp@math.pku.edu.cn` (Z. P. Li)

elastic energy density function $W(\nabla \boldsymbol{u}) : M_+^{2\times 2} \to \mathrm{R}^+$ of the material be poly-convex. Since the material is incompressible, the deformation field must satisfy the constraint $\det \nabla \boldsymbol{u} = 1$ $a.e.$ in Ω. In a mixed formulation of nonlinear hyper-elasticity boundary value problems, one considers to solve the saddle point problem

$$(\tilde{\boldsymbol{u}}, \tilde{p}) = \arg \inf_{\boldsymbol{u}\in\mathcal{A}} \sup_{p\in L^2(\Omega)} E(\boldsymbol{u}, p), \tag{1.1}$$

where p is the pressure like Lagrangian multiplier (see [11]), $E(\boldsymbol{u}, p)$ is the Lagrangian functional defined as

$$E(\boldsymbol{u}, p) = \int_\Omega \left(W(\nabla \boldsymbol{u}(\boldsymbol{x})) - p(\det \nabla u - 1)\right) d\boldsymbol{x} - \int_{\partial_N \Omega} \boldsymbol{t} \cdot \boldsymbol{u}\, ds, \tag{1.2}$$

with $\boldsymbol{t}$ the traction imposed on the Neumann boundary $\partial_N\Omega$, and where in (1.1) $\mathcal{A}$ is the set of admissible deformation functions given by

$$\mathcal{A} = \begin{cases} \{\boldsymbol{u} \in W^{1,s}(\Omega;\mathbb{R}^2) \text{ is 1-to-1 a.e.} : \boldsymbol{u}|_{\partial_D\Omega} = \boldsymbol{u}_0,\}, & \text{if } \partial_D\Omega \neq \emptyset, \\ \{\boldsymbol{u} \in W^{1,s}(\Omega;\mathbb{R}^2) \text{ is 1-to-1 a.e.} : \int_\Omega \boldsymbol{u}\, d\boldsymbol{x} = \boldsymbol{0},\}, & \text{otherwise.} \end{cases} \tag{1.3}$$

Here $s > 1$ is a given Sobolev index, and $\partial_D\Omega$ is the Dirichlet boundary with its 1-D measure $|\partial_D\Omega| \neq 0$.

The variational form of the Euler-Lagrange equation, i.e., the equilibrium equation, of the mixed formulation (1.1), can be expressed as

$$\begin{cases} \displaystyle\int_\Omega \left(\frac{\partial W(\nabla \boldsymbol{u})}{\partial \nabla \boldsymbol{u}} : \nabla \boldsymbol{v} - p\,(\operatorname{cof} \nabla \boldsymbol{u} : \nabla \boldsymbol{v})\right) d\boldsymbol{x} = \int_{\partial_N\Omega} \boldsymbol{t}\cdot\boldsymbol{v}\, ds, & \forall \boldsymbol{v} \in \mathcal{X}_0, \\ \displaystyle\int_\Omega q\,(\det \nabla \boldsymbol{u} - 1)\; d\boldsymbol{x} = 0, & \forall q \in \mathcal{M}, \end{cases} \tag{1.4}$$

where $\operatorname{cof} \nabla \boldsymbol{u}$ denotes the cofactor matrix of $\nabla \boldsymbol{u}$, and

$$\mathcal{M} := L^2(\Omega), \qquad \mathcal{X}_0 = \begin{cases} \left\{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^2) : \boldsymbol{v}|_{\partial_D\Omega} = \boldsymbol{0}\right\}, & \text{if } \partial_D\Omega \neq \emptyset, \\ \left\{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^2) : \int_\Omega \boldsymbol{v}\, d\boldsymbol{x} = \boldsymbol{0}\right\}, & \text{otherwise,} \end{cases} \tag{1.5}$$

are the test function spaces for the pressure p and deformation $\boldsymbol{u}$ respectively.

In the present paper, based on the variational form of Euler-Lagrange equation (1.4), a mixed finite element method (MFEM), using dual-parametric piecewise bi-quadratic and affine (DP-Q2-P1) finite element approximations for the deformation $\boldsymbol{u}$ and pressure like Lagrangian multiplier p respectively, is developed and analyzed for the numerical computation of incompressible nonlinear elasticity boundary value problems with large deformation gradient, and a damped Newton method is applied to solve the resulted discrete problem. The method is shown to be stable (locking free) under some reasonable assumptions on the mesh regularity (see (M1) and (M2) in Section 2.1), the damping criteria (see (C1) and (C2) in Section 2.2) and the stability