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Robust Globally Divergence-Free Weak Galerkin Finite Element Methods for Unsteady Natural Convection Problems

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Abstract. This paper proposes a class of semi-discrete and fully discrete weak Galerkin finite element methods for unsteady natural convection problems in two and three dimensions. In the space discretization, the methods use piecewise polynomials of degrees k, $k-1$, and k $(k \geq 1)$ for the velocity, pressure and temperature approximations in the interior of elements, respectively, and piecewise polynomials of degree k for the numerical traces of velocity, pressure and temperature on the interfaces of elements. In the temporal discretization of the fully discrete method, the backward Euler difference scheme is adopted. The semi-discrete and fully discrete methods yield globally divergence-free velocity solutions. Wellposedness of the semi-discrete scheme is established and a priori error estimates are derived for both the semi-discrete and fully discrete schemes. Numerical experiments demonstrate the robustness and efficiency of the methods.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

Key words: Unsteady natural-convection, semi-discrete and fully discrete, weak Galerkin method, globally divergence-free, error estimate.

1. Introduction

Let $\Omega = \Omega_f \cup \Omega_s \subset \mathbb{R}^d$ $(d = 2, 3)$ be a polygonal/polyhedral domain with two disjoint polygonal/polyhedral subdomains Ω_f and Ω_s. We consider the following unsteady natural convection (or conduction-convection) problem: seek the velocity $\boldsymbol{u}(\boldsymbol{x}, t) : \Omega \times [0, t^*] \rightarrow \mathbb{R}^d$, the pressure $p(\boldsymbol{x}, t) : \Omega \times [0, t^*] \rightarrow \mathbb{R}$ and the temperature $T(\boldsymbol{x}, t)$:

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$\Omega \times [0, t^*] \to \mathbb{R}$ such that

$$\begin{cases} \boldsymbol{u}_t - \Pr \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p - \Pr Ra \boldsymbol{j} T = \boldsymbol{f} & \text{in } \Omega_f \times [0, t^*], \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega_f \times [0, t^*], \\ T_t - \nabla \cdot (\kappa \nabla T) + (\boldsymbol{u} \cdot \nabla) T = g & \text{in } \Omega \times [0, t^*], \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) & \text{in } \Omega_f, \\ T(\boldsymbol{x}, 0) = T_0(\boldsymbol{x}) & \text{in } \Omega, \\ \boldsymbol{u} \equiv \boldsymbol{0} & \text{in } \Omega_s \cup \partial\Omega_f \times [0, t^*], \\ T = 0 & \text{on } \partial\Omega \times [0, t^*]. \end{cases} \tag{1.1}$$

Here $\Pr$ and Ra denote the Prandtl number and Rayleigh number, respectively. $\kappa > 0$ is the thermal conductivity parameter with $\kappa = \kappa_f$ in Ω_f and $\kappa = \kappa_s$ in Ω_s and k_f and k_s are positive constants. $\boldsymbol{j}$ is the vector of gravitational acceleration with $\boldsymbol{j} = (0,1)^T$ when $d = 2$ and $\boldsymbol{j} = (0,0,1)^T$ when $d = 3$. $\boldsymbol{f}(\boldsymbol{x}, t), g(\boldsymbol{x}, t)$ are the forcing functions. And $\boldsymbol{u}_0(\boldsymbol{x})$ and $T_0(\boldsymbol{x})$ are initial data satisfying $\boldsymbol{u}_0|_{\partial\Omega_f} = 0$, $T_0|_{\partial\Omega} = 0$.

The model problem (1.1), arising both in nature and in engineering applications, is a coupled system of fluid flow, governed by the incompressible Navier-Stokes equations and heat transfer, governed by the energy equation. Natural convection heat transfer in a partially heated enclosure is an important issue due to its wide applications in buildings or cooling of flush mounted electronic heaters. In [3] error analysis were carried out for some semi-discrete finite element methods. A characteristic variational multiscale method was proposed in [1]. In [27] Petrov-Galerkin least squares mixed finite element methods were developed. A modified characteristics Gauge–Uzawa finite element method was presented in [39] which combines the modified characteristics finite element method and the Gauge–Uzawa method. In [40] a divergence-free fully discrete mixed finite element method was given using the Crank–Nicolson extrapolation scheme in the temporal discretization. We also refer to [4, 14, 15, 19, 20, 26, 28–31, 35, 36, 38, 42, 50, 51] for some other developments of efficient numerical methods for the steady and unsteady natural convection problems.

In [18], a class of globally divergence-free weak Galerkin (WG) finite element methods were developed and analyzed for the steady natural convection problems. The WG method was first proposed and analyzed to solve second-order elliptic problems [43, 44]. It is designed by using a weakly defined gradient operator over functions with discontinuity and then allows the use of totally discontinuous functions in the finite element procedure. In [5], a class of robust globally divergence-free weak Galerkin methods for Stokes equations were developed and then were extended in [52] to solve incompressible quasi-Newtonian Stokes equations. We refer to [6–9, 16, 21, 23, 25, 32, 45–49, 53] for some other developments and applications of the WG method. We note that in some special cases, the WG method is equivalent to the hybridized discontinuous Galerkin (HDG) method proposed in [11]; see, e.g., [10, 12, 13, 22, 24, 34] for some related works on the HDG method.

In this paper, we shall consider semi-discrete and fully discrete WG methods based on [18] for the unsteady natural convection problem (1.1). The methods include as unknowns the velocity, pressure and temperature variables both in the interior of ele-