Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2018-0016

Vol. **26**, No. 4, pp. 1008-1038
October 2019

Vector Penalty-Projection Methods for Open Boundary Conditions with Optimal Second-Order Accuracy

Philippe Angot[1] and Rima Cheaytou[1,*]

[1] *Aix-Marseille Université, Institut de Mathématiques de Marseille (I2M) – CNRS UMR7373, Centrale Marseille – France.*

Communicated by Kun Xu

Received 17 January 2018; Accepted (in revised version) 11 December 2018

Abstract. Recently, a new family of splitting methods, the so-called vector penalty-projection methods (VPP) were introduced by Angot et al. [4, 7] to compute the solution of unsteady incompressible fluid flows and to overcome most of the drawbacks of the usual incremental projection methods. Two different parameters are related to the VPP methods: the augmentation parameter $r \geq 0$ and the penalty parameter $0 < \varepsilon \leq 1$. In this paper, we deal with the time-dependent incompressible Stokes equations with open boundary conditions using the VPP methods. The spatial discretization is based on the finite volume scheme on a Marker and Cells (MAC) staggered grid. Furthermore, two different second-order time discretization schemes are investigated: the second-order Backward Difference Formula (BDF2) known also as Gear's scheme and the Crank-Nicolson scheme. We show that the VPP methods provide a second-order convergence rate for both velocity and pressure in space and time even in the presence of open boundary conditions with small values of the augmentation parameter r typically $0 \leq r \leq 1$ and a penalty parameter ε small enough typically $\varepsilon = 10^{-10}$. The resulting constraint on the discrete divergence of velocity is not exactly equal to zero but is satisfied approximately as $\mathcal{O}(\varepsilon \delta t)$ where ε is the penalty parameter (taken as small as desired) and δt is the time step. The choice $r = 0$ requires special attention to avoid the accumulation of the round-off errors for very small values of ε. Indeed, it is important in this case to directly correct the pressure gradient by taking account of the velocity correction issued from the vector penalty-projection step. Finally, the efficiency and the second-order accuracy of the method are illustrated by several numerical test cases including homogeneous or non-homogeneous given traction on the boundary.

AMS subject classifications: 76D05, 76D07, 35Q35

Key words: Vector penalty-projection methods, Navier-Stokes equations, incompressible viscous flows, open boundary conditions, second-order accuracy.

*Corresponding author. *Email addresses:* philippe.angot@univ-amu.fr (P. Angot), rima.cheaytou@gmail.com (R. Cheaytou)

1 Introduction

The numerical solution of incompressible flows has always been an important subject in fluid dynamics. The major difficulty in numerically solving unsteady incompressible Navier-Stokes equations in primitive variable form arises from the fact that the velocity and the pressure are coupled by the incompressibility constraint at each time step. There are numerous ways to discretize these equations, see e.g., the short review in [4]. Undoubtedly, the most popular are operator-splitting discretization schemes known as projection methods. This family of methods has been introduced by Chorin (1968) and Temam (1969) [14,36]. The interest in projection methods arises from the fact that the computations of the velocity and the pressure are decoupled by a two-step predictor-corrector procedure which significantly reduces the computational cost. In the first step, an intermediate velocity field is computed by solving momentum equations, ignoring the incompressibility constraint. In the second step, the predicted velocity field is projected onto a divergence-free vector field in order to get the pressure and the corrected velocity that satisfies the mass equation using the Helmholtz-Hodge decomposition. However, this process introduces a new numerical error, often named the splitting error, which must be at worst of the same order as the time discretization error. These projection methods were improved by Goda [18] in 1979 and named "the standard incremental projection methods"; they were popularized by Van Kan [38] in 1986 who introduced a second-order incremental pressure-correction scheme. It is well-known that in the projection step, a difficulty arises from the existence of an artificial pressure Neumann boundary condition which spoils the numerical solution of the pressure. This phenomenon was corrected by a variant proposed by Timmermans et al. [37] and analyzed by Guermond et al. [19] under the name "rotational incremental projection methods". A series of fractional step techniques including pressure-correction and incremental projection methods can be found in the review paper of Guermond et al. [20]. In 1992, Shen [35] introduced a modified approach which consists in adding a penalty term built from the divergence constraint in the first step of the scheme of the same form as in Augmented Lagrangian methods [17]. This approach is called "penalty-projection method". The same idea was suggested independently by Caltagirone and Breil [12] with some additional variants and was called "vector-projection step". In the same way, Jobelin et al. [30] proposed a numerical scheme which falls in the category of the penalty-projection method. This scheme generalizes the prediction step by an augmentation parameter totally independent of the time step and modifies consistently the projection step; numerical results using finite element approximation show that only small or moderate values of the augmentation parameter r are sufficient to get accurate results. This numerical scheme was also theoretically analyzed in [35] and later in [8].

Recently, a new family of methods, the so-called "vector penalty-projection methods" (VPP) was proposed in [4]. Two parameters are related to the VPP methods: the augmentation parameter $r>0$ and the penalty-parameter $0<\varepsilon\leq 1$. These methods represent a compromise between the best properties of both classes: the Augmented Lagrangian