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WONG–ZAKAI APPROXIMATIONS OF STOCHASTIC ALLEN–CAHN EQUATION

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Abstract. We establish an unconditional and optimal strong convergence rate of Wong–Zakai type approximations in Banach space norm for a parabolic stochastic partial differential equation with monotone drift, including the stochastic Allen–Cahn equation, driven by an additive Brownian sheet. The key ingredient in the analysis is the full use of additive nature of the noise and monotonicity of the drift to derive a priori estimation for the solution of this equation. Then we use the factorization method and stochastic calculus in martingale type 2 Banach spaces to deduce sharp error estimation between the exact and approximate Ornstein–Uhlenbeck processes, in Banach space norm. Finally, we combine this error estimation with the aforementioned a priori estimation to deduce the desired strong convergence rate of Wong–Zakai type approximations.

Key words. Stochastic Allen–Cahn equation, Wong–Zakai approximations, strong convergence rate.

1. Introduction

Consider the following parabolic stochastic partial differential equation (SPDE) driven by an additive Brownian sheet W:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x)) + \frac{\partial^2 W(t,x)}{\partial t \partial x}, \quad (t,x) \in (0,T] \times (0,1), \tag{1}$$

with the following initial value and homogeneous Dirichlet boundary conditions:

$$u(t,0) = u(t,1) = 0, \; u(0,x) = u_0(x), \quad (t,x) \in [0,T] \times (0,1). \tag{2}$$

Here f satisfies certain monotone and polynomial growth conditions (see Assumption 2.1). We remark that if $f(x) = x - x^3$, then Eq. (1) is called the stochastic Allen–Cahn equation. This type of stochastic equation, arising from phase transition in materials science by stochastic perturbation such as impurities of the materials, has been extensively studied in the literatures; see, e.g., [4, 12] for one-dimensional white noises and [9, 13] for possibly high-dimensional colored noises.

The main concern in this paper is to derive an unconditional and optimal strong convergence rate of Wong–Zakai–Galerkin approximations to simulate the Brownian sheet in Eq. (1). Specifically, we simulate the space-time white noise by temporal piecewise constant approximation and then make the spectral projection to this temporal approximation (see Eq. (19)). This type of approximation and its versions, such as the spatiotemporal piecewise constant approximation, have been investigated by many researchers in mathematical and numerical settings. See, e.g., [1, 8] for mathematical applications to support theorem in Hölder norm for parabolic SPDEs and the existence of stochastic flow for a stochastic differential equation without Lipschitz conditions; see, e.g., [3, 7, 16] for numerical applications to construct Galerkin approximations for SPDEs with Lipschitz coefficients and the

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convergence of the Wong–Zakai approximate attractors to the original attractor of stochastic reaction-diffusion equations.

We note that the same simulation method had been used in [10] for the stochastic Burgers equation, where the authors derived the strong convergence of the proposed simulation method without any algebraic rate. On the other hand, the authors of [12] regularized the white noise by a spatiotemporal Wong–Zakai approximations and apply to a practical Monte–Carlo method combined with an Euler–Galerkin scheme for the stochastic Allen–Cahn equation. They used a probabilistic maximum principle which leads to the assumption that $u_0 \in L^\infty(0,1)$ to prove the conditional convergence rate

$$\Big(\mathbb{E}\Big[\chi_{\Omega_{\tau,h}}\|u-\widehat{u}\|^2_{L^2((0,T)\times(0,1))}\Big]\Big)^{\frac{1}{2}} = \mathcal{O}\Big(\tau^{\frac{1}{4}} + h/\tau^{\frac{1}{4}}\Big),$$

in a large subset $\Omega_{\tau,h} \subset \Omega$ such that $\mathbb{P}(\Omega_{\tau,h}) \to 1$ as the temporal and spatial step sizes τ, h tend to 0, where u and $\widehat{u}$ denote the exact and Wong–Zakai approximate solution of the stochastic Allen–Cahn equation, respectively.

These problems are main motivations for this study to give an unconditional and optimal strong convergence rate of Wong–Zakai-type approximations of Eq. (1) with a monotone drift which grows polynomially. Our approach shows that, to derive a strong convergence rate of the proposed Wong–Zakai–Galerkin approximations under the $L^\infty(0,T;L^2(\Omega;L^2(0,1)))$-norm, it is necessary to bound the exact solution and derive the strong convergence rate of the associated exact and approximate Ornstein–Uhlenbeck processes in the $L^p(0,T;L^p(\Omega;L^p(0,1)))$-norm and the $L^\infty(0,T;L^l(\Omega;L^l(0,1)))$-norm, respectively, for possibly large indices $p, l > 2$ (see (26)). This is mainly due to the appearance of the polynomial growth in the nonlinearity and quite different from that of [3, 7] where these authors only needed to deal with the $L^\infty(0,T;L^2(\Omega;L^2(0,1)))$-norm.

To derive the aforementioned a priori estimation for the solution of Eq. (1), the key ingredient in our analysis is by making full use of the additive nature of the noise which allows the transformation of Eq. (1) to the equivalent random partial differential equation (PDE) (13) and the monotonicity of f (see Proposition 2.1). Then we combine the factorization method with stochastic calculus in martingale type 2 Banach spaces to bounded uniformly the exact and approximate Ornstein–Uhlenbeck processes and derive a sharp strong convergence rate for them in Banach setting (see Lemma 2.1 and Theorem 3.1).

The main result is the following unconditional strong convergence rate of the aforementioned Wong–Zakai–Galerkin approximations applied to Eq (1):

$$\sup_{t\in[0,T]} \Big(\mathbb{E}\Big[\|u(t)-u^{m,n}(t)\|^p_{L^p(0,1)}\Big]\Big)^{\frac{1}{p}} = \mathcal{O}\Big[\Big(\frac{1}{m}\Big)^{\frac{1}{4}} \wedge \Big(\frac{1}{n}\Big)^{\frac{1}{2}}\Big], \tag{3}$$

for any $1 \le p < \frac{p_*}{2} + 1$ provided that $u_0 \in L^{p_*}(\Omega; L^{p_*}(0,1))$ (see Theorem 3.2 and Remark 3.2). Here m, n are the number of temporal steps and dimension of spectral Galerkin space, and u and $u^{m,n}$ denote the exact solution of Eq. (1) and the Wong–Zakai–Galerkin approximate solution of Eq. (19), respectively. Note that we generalize, in a separate paper [14], the approach of the present paper in combination with new techniques to derive a strong convergence rate of a fully discrete approximation for Eq. (1) under certain regularity condition on the initial datum.