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DISCONTINUOUS GALERKIN METHODS FOR THE BIHARMONIC PROBLEM ON POLYGONAL AND POLYHEDRAL MESHES

ZHAONAN DONG

Abstract. We introduce an *hp*-version symmetric interior penalty discontinuous Galerkin finite element method (DGFEM) for the numerical approximation of the biharmonic equation on general computational meshes consisting of polygonal/polyhedral (polytopic) elements. In particular, the stability and *hp*-version a-priori error bound are derived based on the specific choice of the interior penalty parameters which allows for edges/faces degeneration. Furthermore, by deriving a new inverse inequality for a special class of polynomial functions (harmonic polynomials), the proposed DGFEM is proven to be stable to incorporate very general polygonal/polyhedral elements with an *arbitrary* number of faces for polynomial basis with degree $p = 2, 3$. The key feature of the proposed method is that it employs elemental polynomial bases of total degree $\mathcal{P}_p$, defined in the physical coordinate system, without requiring the mapping from a given reference or canonical frame. A series of numerical experiments are presented to demonstrate the performance of the proposed DGFEM on general polygonal/polyhedral meshes.

Key words. Discontinuous Galerkin, polygonal/polyhedral elements, inverse estimates, biharmonic problems.

1. Introduction

Fourth-order boundary-value problems have been widely used in mathematical models from different disciplines, see [23]. The classical conforming finite element methods (FEMs) for the numerical solution of the biharmonic equation require that the approximate solution lie in a finite-dimensional subspace of the Sobolev space $H^2(\Omega)$. In particular, this necessitates the use of C^1 finite elements, such as Argyris elements. In general, the implementation of C^1 elements is far from trivial. To relax the C^1 continuity requirements across the element interfaces, nonconforming FEMs have been commonly used by engineers and also analysed by mathematicians; we refer to the monograph [16] for the details of above mentioned FEMs. For a more recent approach, we mention the C^0 interior penalty methods, see [21, 10] for details. Another approach to avoid using C^1 elements is to use the mixed finite element methods, we refer to the monograph [8] and the reference therein.

In the last two decades, discontinuous Galerkin FEMs (DGFEMs) have been considerably developed as flexible and efficient discretizations for a large class of problems ranging from computational fluid dynamics to computational mechanics and electromagnetic theory. In the pioneer work [6], DGFEMs were first introduced as a special class of nonconforming FEMs to solve the biharmonic equation. For the overview of the historical development of DGFEMs, we refer to the important paper [5] and monographs [17, 18] and all the reference therein. DGFEMs are attractive as they employ the discontinuous finite element spaces, giving great flexibility in the design of meshes and polynomial bases, providing general framework for *hp*-adaptivity. For the biharmonic problem, *hp*-version interior penalty (IP) DGFEMs were introduced in [30, 31, 34]. The stability of different IP-DGFEMs and a priori

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error analysis in various norms have been studied in those work. Additionally, the exponential convergence for the p-version IP-DGFEMs was proven in [24]. The a posterior error analysis of the symmetric IP-DGFEM has been done in [25]. In [22, 28], the domain decomposition preconditioners have been designed for IP-DGFEMs.

More recently, DGFEMs on meshes consisting of general polygons in two dimensions or general polyhedra in three dimensions, henceforth termed collectively as *polytopic*, have been proposed [2, 15, 13, 12, 7, 1]. The key interest of employing polytopic meshes is predominant by the potential reduction in the total numerical degrees of freedom required for the numerical solution of PDE problems, which is particularly important in designing the adaptive computations for PDE problems on domains with micro-structures. Hence, polytopic meshes can naturally be combined with DGFEMs due to their element-wise discontinuous approximation. In our works [15, 13], an hp-version symmetric IP-DGFEM was introduced for the linear elliptic problem and the general advection-diffusion-reaction problem on meshes consisting of d-dimensional polytopic elements were analysed. The key aspect of the method is that the DGFEM is stable on general polytopic elements in the presence of degenerating $(d-k)$-dimensional element facets, $k = 1, \dots, d-1$, where d denotes the spatial dimension. The main mesh assumption for the polytopic elements is that all the elements have a uniformly bounded number of $(d-1)$-dimensional faces, without imposing any assumptions on the measure of faces. (Assumption 4.1 in this work) In our work [12], we proved that the IP-DGFEM is stable for second order elliptic problem on polytopic elements with arbitrary number of $(d-1)$-dimensional faces, without imposing any assumptions on the measure of faces. The mesh assumption for the polytopic elements is that all the elements should satisfy a shape-regular condition, without imposing any assumptions on the measure of faces or number of faces (Assumption 5.1 in this work). For details of DGFEMs on polytopic elements, we refer to the monograph [14].

To support such general element shapes, without destroying the local approximation properties of the DGFEM developed in [15, 13, 12], polynomial spaces defined in the physical frame, rather than mapped polynomials from a reference element, are typically employed. It has been demonstrated numerically that the DGFEM employing $\mathcal{P}_p$-type basis achieves a faster rate of convergence, with respect to the number of degrees of freedom presented in the underlying finite element space, as the polynomial degree p increases, for a given fixed mesh, than the respective DGFEM employing a (mapped) $\mathcal{Q}_p$ basis on tensor-product elements; we refer [19] for more numerical examples. The proof of the above numerical observations is given in [20].

In this work, we will extend the results in [15, 13, 12] to cover hp-version IP-DGFEMs for biharmonic PDE problems. We will prove the stability and derive the a priori error bound for the hp-version IP-DGFEM on general polytopic elements with possibly degenerating $(d-k)$-dimensional facets, under two different mesh assumptions. (Assumption 4.1 and 5.1). The key technical difficulty is that the H^1-seminorm to L_2-norm inverse inequality for general polynomial functions defined on polytopic elements with arbitrary number of faces is empty in the literature. To address this issue, we prove a new inverse inequality for *harmonic polynomial functions* on polytopic elements satisfying Assumption 5.1. With the help of the new inverse inequality, we prove the stability and derive the a priori error bound for the proposed DGFEM employing $\mathcal{P}_p$ basis, $p = 2, 3$, under the Assumption 5.1. Here, we mention that there already exist different polygonal discretization methods