J. Math. Study
doi: 10.4208/jms.v52n3.19.06

Vol. **52**, No. 3, pp. 320-340
September 2019

A Domain Decomposition Method for Linearized Boussinesq-Type Equations

Joao Guilherme Caldas Steinstraesser[1,3], Gaspard Kemlin[2] and Antoine Rousseau[1,*]

[1] *Inria, IMAG, Univ Montpellier, CNRS, Montpellier, France.*
[2] *Inria Chile, Avenida Apoquindo 2827, Las Condes, Santiago, Chile.*
[3] *MERIC, Avenida Apoquindo 2827, Las Condes, Santiago, Chile.*

Received January 7, 2019; Accepted May 26, 2019

Abstract. In this paper, we derive discrete transparent boundary conditions for a class of linearized Boussinesq equations. These conditions happen to be non-local in time and we test numerically their accuracy with a Crank-Nicolson time-discretization on a staggered grid. We use the derived transparent boundary conditions as interface conditions in a domain decomposition method, where they become local in time. We analyze numerically their efficiency thanks to comparisons made with other interface conditions.

AMS subject classifications: 65M55

Key words: Boussinesq-type equations, finite differences scheme, transparent boundary conditions, domain decomposition, interface conditions, Schwarz alternating method.

1 Introduction

Among the main challenges faced in the mathematical framework of coastal engineering is the study of wave propagation in the nearshore area. One field of research in this topic makes use of the Boussinesq equations for water of varying depth that describe the nonlinear propagation of waves in shallow water. The work of Peregrine [1], Green and Naghdi [2] laid the basis for many Boussinesq-type equations used nowadays. The dispersion properties of these equations have then been improved by Nwogu [3] for practical numerical simulation of ocean wave processes from deep to shallow water. In this paper, we work on the equations derived by Nwogu [3] and that can be recalled as follows. Consider a three-dimensional wave field with surface elevation $\eta(x,y,t)$ over a non-constant water depth $h(x,y)$ and with speed $\boldsymbol{u}(x,y,z,t)=(u,v)$, respectively the

*Corresponding author. *Email addresses:* `joao-guilherme.caldas-steinstraesser@inria.fr` (J.G. Caldas Steinstraesser), `gaspard.kemlin@inria.fr` (G. Kemlin), `antoine.rousseau@inria.fr` (A. Rousseau)

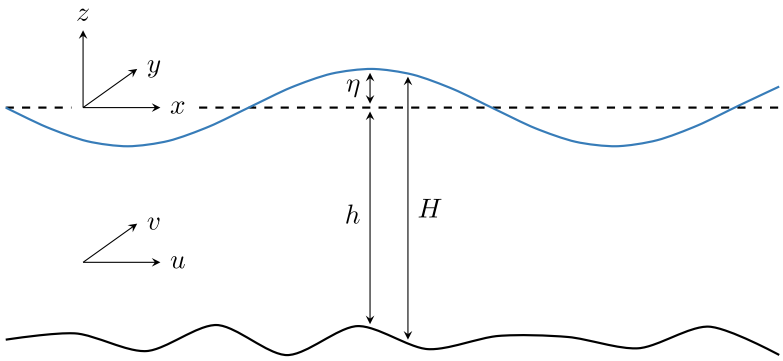

Figure 1: Definition of the quantities η, h, H, $\boldsymbol{u}$.

speeds along the x and y axis, defined at a reference depth $z = z(x,y)$. Figure 1 provides a sketch where we represented these quantities. With $\boldsymbol{\tau}$ being the bottom shear stress, Nwogu [3] obtained (1.1), which consists of a continuity equation and a momentum equation. These equations have been used in a `C/Matlab` program known as the *Boussinesq Ocean and Surf Zone* (BOSZ) model, developed by Roeber and Cheung [4].

$$\begin{cases} \eta_t + \nabla\cdot[(h+\eta)\boldsymbol{u}] + \nabla\cdot\left[\left(\dfrac{z^2}{2}-\dfrac{h^2}{6}\right)h\nabla(\nabla\cdot\boldsymbol{u}) + \left(z+\dfrac{h}{2}\right)h\nabla(\nabla\cdot(h\boldsymbol{u}))\right] = 0, \\ \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + g\nabla\eta + z\left[\dfrac{z}{2}\nabla(\nabla\cdot\boldsymbol{u}_t) + \nabla(\nabla\cdot(h\boldsymbol{u}_t))\right] + \boldsymbol{\tau} = 0. \end{cases} \tag{1.1}$$

To simplify the framework, we consider in this paper the 1D equations and ignore the bottom shear stress $\boldsymbol{\tau}$. We also consider a constant flat bottom $h = h_0$. Thus, the total height H can be recovered by the relation $H = h_0 + \eta$. Then, we perform a linearization around the equilibrium point $(\bar{\eta},\bar{u}) = (0,\bar{u})$, with $\bar{u} \in \mathbb{R}$, to get

$$\begin{cases} \eta_t + \bar{u}\eta_x + h_0 u_x + \tilde{h} u_{xxx} = 0, \\ u_t + g\eta_x + \bar{u}u_x + \bar{h} u_{xxt} = 0, \end{cases} \tag{1.2}$$

to which we now refer as the *linearized Boussinesq equations*. Here, $\tilde{h}$ and $\bar{h}$ are constants defined by

$$\tilde{h} = \left(\frac{z^2}{2} + h_0 z + \frac{h_0^2}{3}\right)h_0, \qquad \bar{h} = z\left(\frac{z}{2} + h_0\right). \tag{1.3}$$

Using the value $z = -0.53753 \times h_0$ (see [3]) and $h_0 = 1$, we have that $\bar{h}$ and $\tilde{h}$ are both negative. Setting $\bar{u} = 0$, $\tilde{h} = 0$, $\bar{h} = -\varepsilon$ a small parameter, $g = 1$ and $h_0 = 1$ leads to the formulation of the linearized Green-Naghdi equations, for which discrete transparent boundary conditions have been derived by Kazakova and Noble [5]. In this paper, we focus on the case where $\bar{u} = 0$ for the sake of simplicity, but the conclusions of Section 3