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Vector-valued Inequalities for Commutators of Singular Integrals on Herz Spaces with Variable Exponents

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Abstract: Based on the theory of variable exponents and BMO norms, we prove the vector-valued inequalities for commutators of singular integrals on both homogeneous and inhomogeneous Herz spaces where the two main indices are variable exponents. Furthermore, we show that a wide class of commutators generated by BMO functions and sublinear operators satisfy vector-valued inequalities.

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1 Introduction

The modern development of variable exponent function spaces was initiated by Kováčik and Rákosník[1] appearing in 1991, and was originally tied closely to the theory of fluid dynamics, image restoration and PDE with non-standard growth conditions, for an overview we refer to [2]–[4]. On the other hand, function spaces with variable exponents have many properties in common with the classical cases, but they also differ in surprising and subtle ways. For instance, the variable exponent Lebesgue spaces are not translation invariant. More precisely, if $p(\cdot)$ is non-constant in $\mathbf{R}^n$, then there always exist $f \in L^{p(\cdot)}(\mathbf{R}^n)$ and $h \in \mathbf{R}^n$ such that $f(x+h)$ is not in $L^{p(\cdot)}(\mathbf{R}^n)$. As a consequence $L^{p(\cdot)}(\mathbf{R}^n)$ spaces are not rearrangement invariant Banach function spaces, and so a great deal of classical machinery is not applicable (see [5]). For this reason, apart from useful application considerations, the

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motivation to study such spaces has an intrinsic interest. In the past 25 years, especially the past decade, we have witnessed a rapid growth in the study of these and related spaces, see [6]–[13] and references therein.

The classical Herz spaces have been playing an integral role in harmonic analysis and PDE. After they were introduced in [14], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they are good substitutes of the ordinary Hardy spaces when considering the boundedness of non-translation invariant singular integral operators. They also appear in the summability of Fourier transforms and in the regularity theory for elliptic and parabolic equations in divergence form, see [15]–[18] for example. Recently, Izuki[7] defined homogeneous Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbf{R}^n)$ and inhomogeneous Herz spaces $K^{\alpha,q}_{p(\cdot)}(\mathbf{R}^n)$ with variable exponent $p(\cdot)$ but fixed $\alpha \in \mathbf{R}$ and proved the boundedness of sublinear operators and commutators of singular integrals on those spaces. Somewhat later, Almeida and Drihem[6] introduced the generalized Herz spaces $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbf{R}^n)$ and $K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbf{R}^n)$, where the exponent α is variable as well. Under natural regularity assumptions on the exponent functions, they showed that many classical operators, such as maximal, potential and Calderón-Zygmund operators, are bounded on such spaces.

Given a locally integrable function K defined on $\mathbf{R}^n\backslash\{0\}$, suppose that the Fourier transform of K is bounded, and K satisfies

$$|K(x)| \lesssim |x|^{-n}, \qquad |\nabla K(x)| \lesssim |x|^{-n-1}. \tag{1.1}$$

Then the singular integral operator $\widetilde{T}$ defined by $\widetilde{T}f(x) = K * f(x)$ is a bounded operator on the classical Lebesgue spaces $L^p(\mathbf{R}^n)$. Let $b \in \mathrm{BMO}(\mathbf{R}^n)$. Define the commutator $[b, \widetilde{T}]$ to be the operator

$$[b, \widetilde{T}]f(x) := b(x)\widetilde{T}f(x) - \widetilde{T}(bf)(x). \tag{1.2}$$

A celebrated result of Coifman*et al.*[19] states that the operator $[b, \widetilde{T}]$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. Moreover, this commutator has proved to be of interest in many contexts and in particular in the theory of PDE, we shall only mention the recent results in the theory of elliptic equations with discontinuous coefficients (see [20] and [21]).

In [22], Cruz-Uribe *et al.* showed that $\widetilde{T}$ and $[b, \widetilde{T}]$ are both bounded operators on the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbf{R}^n)$. Using the theory of weighted norm inequalities and extrapolation, they also proved some vector-valued inequalities for such operators. A further step was taken by Izuki[23], who established the similar vector-valued estimates for the commutator $[b, \widetilde{T}]$ on Herz spaces $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbf{R}^n)$ and $K^{\alpha,q}_{p(\cdot)}(\mathbf{R}^n)$. From Remark 2.2 below, one can see that if $\alpha(\cdot)$ is constant, then $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbf{R}^n)$ and $K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbf{R}^n)$ coincide with $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbf{R}^n)$ and $K^{\alpha,q}_{p(\cdot)}(\mathbf{R}^n)$, respectively. Hence, it is of interest to ask whether the vector-valued inequalities still hold for $[b, \widetilde{T}]$ on $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}(\mathbf{R}^n)$ and $K^{\alpha(\cdot),q}_{p(\cdot)}(\mathbf{R}^n)$? Our first aim in this paper is to give an affirmative answer to the above question.

On the other hand, one of the important problems on Herz spaces is boundedness of sublinear operators satisfying the size condition

$$|Tf(x)| \lesssim \int_{\mathbf{R}^n} |x-y|^{-n}|f(y)|\mathrm{d}y, \qquad x \notin \mathrm{supp} f, \tag{1.3}$$