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The Direct Sum Decomposition of Type G_2 Lie Algebra

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Abstract: This article mainly discusses the direct sum decomposition of type G_2 Lie algebra, which, under such decomposition, is decomposed into a type A_1 simple Lie algebra and one of its modules. Four theorems are given to describe this module, which could be the direct sum of two or three irreducible modules, or the direct sum of weight modules and trivial modules, or the highest weight module.

Key words: simple Lie algebra G_2, simple Lie algebra A_1, direct sum decomposition, the highest weight module

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1 Introduction and Main Results

The classification of Lie algebra is one of traditional fields of research in [1]–[3]. However, the detailed description of structures of basic 9 types of simple Lie algebra is not easy to achieve in [1]. This article tries to analysis the structure of type G_2 Lie algebra with the help of direct sum decomposition. This method involves the simple type A_1 Lie algebra, the irreducible modules, the trivial modules, the weight modules and the highest weight module. Generally, it is difficult to analysis the structure of type G_2 Lie algebra precisely only by definitions, properties and theorems of Lie algebra. But using the direct sum decomposition enables us to get the detailed description of the structure of type G_2 Lie algebra. The main result is given by the following theorem.

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Theorem 1.1 *The simple type G_2 Lie algebra can be direct sum decomposed as $G_2 = A_1 \oplus M$. There are four possibilities for M:*

(1) *M is the direct sum of four two-dimensional weight modules and three trivial modules*;

(2) *M is the highest weight module and the highest weight is* 10;

(3) *M is the direct sum of two irreducible modules*;

(4) *M is the direct sum of three irreducible modules.*

This article begins with the discussion of the background in [1], [4]–[6] and detailed information of type G_2 Lie algebra in [7]–[11], which could help to understand our work.

2 Basic Description

We begin this section by giving some useful result from [1].

Assume that the standard orthogonal basis of $\mathbf{R}^3$ is $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\varepsilon}_3$, and E is the plane in $\mathbf{R}^3$ which goes through the origin and is orthogonal to the vector $\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_3$. Note I as the glen Z-span by $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\varepsilon}_3$, where $I' = I \bigcap E$, and root system

$$\varPhi = \pm\{\boldsymbol{\alpha} \in I',\ (\boldsymbol{\alpha},\, \boldsymbol{\alpha}) = 2 \text{ or } 6\}.$$

Then the irreducible root system of a simple type G_2 Lie algebras is

$$\varPhi = \pm\{\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2,\, \boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_3,\, \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_3,\, 2\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_3,\, 2\boldsymbol{\varepsilon}_2 - \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_3,\, 3\boldsymbol{\varepsilon}_3 - \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\}.$$

The simple root for simple type G_2 Lie algebra can be chosen for $\boldsymbol{\alpha}_1 = \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2$, $\boldsymbol{\alpha}_2 = -2\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_3$. The highest root is $3\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2$, and the shortest root is $2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$.

If the base is fix to be $\varDelta = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$, then the strongly dominant is: $\boldsymbol{\lambda}_1 = 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$, $\boldsymbol{\lambda}_2 = 3\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2$. This has a 1-1 irreducible representation of a 7×7 matrix. And $\dim G_2 = 14$, $\boldsymbol{d}_i = \boldsymbol{e}_{i+1,i+1} - \boldsymbol{e}_{i+4,i+4}$ $(i = 1, 2, 3)$. The Cartan Subalgebra (CSA for short) of G_2 is H, $H = \{\boldsymbol{h}_1,\, \boldsymbol{h}_2 \mid \boldsymbol{h}_1 = \boldsymbol{d}_1 - \boldsymbol{d}_2,\, \boldsymbol{h}_2 = \boldsymbol{d}_2 - \boldsymbol{d}_3\}$, $\dim H = 2$.

The six long roots about H is $\boldsymbol{g}_{i,-j}$ $(i \neq j,\ i, j = 1, 2, 3)$,

$$\boldsymbol{g}_{1,-2} = \boldsymbol{g}_{2,-1} = \boldsymbol{e}_{23} - \boldsymbol{e}_{65},$$
$$\boldsymbol{g}_{1,-3} = \boldsymbol{g}_{3,-1} = \boldsymbol{e}_{24} - \boldsymbol{e}_{75},$$
$$\boldsymbol{g}_{2,-3} = \boldsymbol{g}_{3,-2} = \boldsymbol{e}_{34} - \boldsymbol{e}_{76}.$$

The six short roots about H is $\boldsymbol{g}_{\pm i}$ $(i = 1, 2, 3)$,

$$\boldsymbol{g}_1 = -\boldsymbol{g}_{-1}^t = \sqrt{\boldsymbol{e}_{12} - \boldsymbol{e}_{51}} - \sqrt{\boldsymbol{e}_{37} - \boldsymbol{e}_{46}},$$
$$\boldsymbol{g}_2 = -\boldsymbol{g}_{-2}^t = \sqrt{\boldsymbol{e}_{13} - \boldsymbol{e}_{61}} - \sqrt{\boldsymbol{e}_{27} - \boldsymbol{e}_{45}},$$
$$\boldsymbol{g}_3 = -\boldsymbol{g}_{-3}^t = \sqrt{\boldsymbol{e}_{14} - \boldsymbol{e}_{71}} - \sqrt{\boldsymbol{e}_{26} - \boldsymbol{e}_{35}}.$$

The 12 roots of the above are the common feature vectors of $\mathrm{ad}H$.

The operations between the bases of the G_2

$$[\boldsymbol{g}_{i,-j},\, \boldsymbol{g}_{k,-l}] = \delta_{jk}\boldsymbol{g}_{i,-l} - \delta_{il}\boldsymbol{g}_{k,-j},$$
$$[\boldsymbol{g}_i,\, \boldsymbol{g}_{-i}] = 3\boldsymbol{d}_i - (\boldsymbol{d}_1 + \boldsymbol{d}_2 + \boldsymbol{d}_3),$$
$$[\boldsymbol{g}_{i,-j},\, \boldsymbol{g}_{-k}] = -\delta_{jk}\boldsymbol{g}_{-i},$$
$$[\boldsymbol{g}_{i,-j},\, \boldsymbol{g}_k] = -\delta_{ik}\boldsymbol{g}_j,$$