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Streamline Diffusion Virtual Element Method for Convection-Dominated Diffusion Problems

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Abstract. A novel streamline diffusion form of virtual element method for convection-dominated diffusion problems is studied. The main feature of the method is that the test function in the stabilised term has the adjoint operator-like form $(-\nabla\cdot(K(\mathbf{x})\nabla v)-\mathbf{b}(\mathbf{x})\cdot\nabla v)$. Unlike the standard VEM, the stabilisation scheme can efficiently avoid nonphysical oscillations. The well-posedness of the problem is also proven and error estimates are provided. Numerical examples show the stability of the method for very large Péclet numbers and its applicability to boundary layer problem.

AMS subject classifications: 65N12, 76R99

Key words: Convection dominated diffusion problem, streamline diffusion virtual element method, stabilisation, optimal convergence, boundary layer problem.

1. Introduction

Diffusion problems play an important role in various applications, including river and air pollution, fluid flow and fluid heat conduction. It is well known that for convection fields, which are relatively large with respect to the diffusivity, the solutions of the corresponding diffusion problems may have boundary and interior layers. Since standard numerical methods do not work well in such situations, a variety of robust schemes for convection-dominated problems, such as streamline upwind/Petrov-Galerkin formulation [15,18,27], residual-free bubbles methods [12,19], discontinuous Galerkin methods [20], finite volume method [25], characteristic finite difference method [26], and multiquadric RBF-FD method [22] have been developed. Recently, Duan *et al.* [17] introduced a numerical method with deterministic and explicit stabilisation parameter for the reaction-convection-diffusion equations having a large reaction coefficient. For equations with arbitrary magnitude of reaction and diffusion, Hsieh and Yang [21] proposed a new method with a test

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function in the stabilising term. We also note the virtual element method (VEM) in [1,3–5]. It can be viewed as the evolution of the mimetic finite difference (MFD) method [13, 23] or as an extension of the traditional finite element method (FEM) to polygons or polyhedron elements [28]. VEM was applied to various problems, such as plate bending [14], elasticity [6], Stokes [2] and Navier-Stokes equations [7], simplified friction problem [29]. The method performs well in geometrically complex domains [8] and with badly shaped polygonal elements [10]. Cangiani *et al.* [16] first discussed a non-consistent SUPG-VEM formulation for the convection-dominated diffusion problem, Benedetto *et al.* [9] established a consistent VEM-SUPG formulation, and Berrone *et al.* [11] developed a SUPG stabilisation for the nonconforming VEM. The robustness of a priori error estimates for these methods can be proved for high Péclet numbers. This shows the efficiency of the SUPG stabilisation.

The present work is motivated by the ideas of [21] and aims to construct a new kind of streamline diffusion VEM for convection-dominated diffusion problems. We use a test function in the stabilisation term in the adjoint-operator-like form $-\nabla \cdot (K\nabla v) - \mathbf{b} \cdot \nabla v$. In contrast to others SUPG formulations, the stabilisation parameter τ has a simple representation and can be easily computed for each element. In order to comply to VEMs with implicit basis functions, we compute the projections of the stabilisation terms onto polynomial spaces. The well-posedness and optimal error estimates of the stabilisation scheme are established by introducing an error norm. Numerical results show that the stabilisation scheme can efficiently treat boundary layer problems and avoid nonphysical oscillations.

The paper is organised as follows. In Section 2, we introduce a model problem and consider a streamline diffusion virtual element method. A priori error estimates are presented by Theorem 3.2 in Section 3, whereas Section 4 contains some results of numerical tests to support the theoretical findings. Finally, some conclusions are drawn in the last section.

2. Streamline Diffusion VEM for a Model Problem

We consider the streamline diffusion virtual element approximations for the following Dirichlet boundary value convection-dominated diffusion problem:

$$\begin{aligned} -\nabla \cdot (K(\mathbf{x})\nabla u) + \mathbf{b}(\mathbf{x}) \cdot \nabla u = f(\mathbf{x}), \quad & \text{in } \Omega, \\ u(\mathbf{x}) = 0, \quad & \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded convex polygonal domain with the boundary $\partial\Omega$, u the physical quantity of interest, $K(\mathbf{x}) \in L^\infty(\Omega)$ the symmetric diffusion tensor, $\mathbf{b}(\mathbf{x}) \in (C(\Omega))^2$ the convection field such that $\nabla \cdot \mathbf{b}(\mathbf{x}) = 0$ in Ω and $f(\mathbf{x})$ a given source function.

Let $(\cdot,\cdot)$ and $\|\cdot\|$ refer to $L^2(\Omega)$ scalar products and norm, respectively, and if D is a subdomain of Ω, we write $(\cdot,\cdot)_D$ and $\|\cdot\|_D$ for the corresponding $L^2(D)$ scalar product and the norm. Besides, $\|\cdot\|_m$ and $|\cdot|_m$ are the $H^m(\Omega)$ norm and semi-norm and $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ the $H^m(D)$ norm and semi-norm. Let $\mathbb{P}_k(D)$ be the space of polynomials of degree at most k on D. The dimension N_P of $\mathbb{P}_k(D)$ is $(1/2)(k+1)(k+2)$.