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COMPUTATIONAL MULTISCALE METHODS FOR LINEAR HETEROGENEOUS POROELASTICITY*

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Abstract

We consider a strongly heterogeneous medium saturated by an incompressible viscous fluid as it appears in geomechanical modeling. This poroelasticity problem suffers from rapidly oscillating material parameters, which calls for a thorough numerical treatment. In this paper, we propose a method based on the local orthogonal decomposition technique and motivated by a similar approach used for linear thermoelasticity. Therein, local corrector problems are constructed in line with the static equations, whereas we propose to consider the full system. This allows to benefit from the given saddle point structure and results in two decoupled corrector problems for the displacement and the pressure. We prove the optimal first-order convergence of this method and verify the result by numerical experiments.

Mathematics subject classification: 65M12, 65M60, 76S05.
Key words: Poroelasticity, Heterogeneous media, Numerical homogenization, Multiscale methods.

1. Introduction

Modeling the deformation of porous media saturated by an incompressible viscous fluid is of great importance for many physical applications such as reservoir engineering in the field of geomechanics [26] or the modeling of the human anatomy for medical applications [6, 22]. To obtain a reasonable model, it is important to couple the flow of the fluid with the behavior of the surrounding solid. Biot proposed a model that couples a Darcy flow with linear elastic behavior of the porous medium [1]. The corresponding analysis was given in [25]. For this so-called poroelastic behavior, pressure and displacement are averaged across (infinitesimal) cubic elements such that pressure and displacement can be treated as variables on the entire domain of interest. Furthermore, the model is assumed to be quasi-static, i.e., an internal equilibrium

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is preserved at any time. In the poroelastic setting, this means that volumetric changes occur slowly enough for the pressure to remain basically constant throughout an infinitesimal element.

If the given material is homogeneous, the poroelastic behavior can be simulated using standard numerical methods such as the finite element method, see for instance [13]. However, if the medium is strongly heterogeneous, the material parameters may oscillate on a fine scale. In such a scenario, the classical finite element method only yields acceptable results if the fine scale is resolved by the spatial discretization, which is unfeasible in practical applications. To overcome this issue, homogenization techniques may be applied, such as the general multiscale finite element method (GMsFEM) [11], used in [4, 5], or the localized orthogonal decomposition technique (LOD) [21] as used in [19] for the similar problem of linear thermoelasticity. The general idea of these methods is to construct low-dimensional finite element spaces which incorporate spatial fine scale features using adapted basis functions. This involves additional computations in the offline stage with the benefit of having much smaller linear systems to solve in every time step due to the much lower amount of degrees of freedom.

In the present paper, a multiscale finite element method is proposed based on the LOD method and adopting the ideas presented in [19]. In contrast to the method of [19], we are able to exploit the saddle point structure of the problem in order to obtain fully symmetric and decoupled corrector problems without the need for additional corrections. Furthermore, this implies that the correctors are independent of the Biot-Willis fluid-solid coupling coefficient, although it may vary rapidly as well.

The work is structured as follows. In Section 2 we present the model problem and introduce the necessary notation. Section 3 is devoted to the discretization of the problem. This includes the classical finite element method on a fine mesh as well as the formulation introduced in [19] translated to poroelasticity. We then introduce the decoupled corrector problems and the resulting new multiscale scheme for which we prove convergence. Finally, numerical results, which illustrate the theoretical findings, are presented in Section 4.

Throughout the paper C denotes a generic constant, independent of spatial discretization parameters and the time step size. Further, $a \lesssim b$ will be used equivalently to $a \leq Cb$.

2. Linear Poroelasticity

2.1. Model problem

We consider the linear poroelasticity problem in a bounded and polyhedral Lipschitz domain $D \subset \mathbb{R}^d$ ($d = 2, 3$) as discussed in [25]. For the sake of simplicity, we restrict ourselves to homogeneous Dirichlet boundary conditions. The extension to Neumann boundary conditions is straightforward. This means that we seek the pressure $p\colon [0,T] \times D \to \mathbb{R}$ and the displacement field $u\colon [0,T] \times D \to \mathbb{R}^d$ within a given time $T > 0$ such that

$$-\nabla \cdot \big(\sigma(u)\big) + \nabla(\alpha p) = 0 \qquad \text{in } (0,T] \times D, \tag{2.1a}$$

$$\partial_t\Big(\alpha \nabla \cdot u + \frac{1}{M} p\Big) - \nabla \cdot \Big(\frac{\kappa}{\nu} \nabla p\Big) = f \quad \text{in } (0,T] \times D \tag{2.1b}$$

with boundary and initial conditions

$$u = 0 \qquad \text{on } (0,T] \times \partial D, \tag{2.1c}$$

$$p = 0 \qquad \text{on } (0,T] \times \partial D, \tag{2.1d}$$

$$p(\cdot, 0) = p^0 \qquad \text{in } D. \tag{2.1e}$$