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Numerical Approaches to Compute Spectra of Non-Self Adjoint Operators and Quadratic Pencils

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Abstract. In this article we are interested in the numerical computation of spectra of non-self adjoint quadratic operators. This leads to solve nonlinear eigenvalue problems. We begin with a review of theoretical results for the spectra of quadratic operators, especially for the Schrödinger pencils. Then we present the numerical methods developed to compute the spectra : spectral methods and finite difference discretization, in infinite or in bounded domains. The numerical results obtained are analyzed and compared with the theoretical results. The main difficulty here is that we have to compute eigenvalues of strongly non-self-adjoint operators which are very unstable.

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1 Introduction

We are interested here in equations like $L(\lambda)u=0$ where $L(\lambda)$ is a linear operator on some linear space $\mathcal{E}$, depending on a complex parameter λ. When $L(\lambda)=L_0-\lambda\mathbb{I}$, this is the usual eigenvalue problem : find $\lambda\in\mathbb{C}$ and $u\in\mathcal{E}$, $u\neq 0$ such that $L(\lambda)u=0$. In many applications, in particular for dissipative problems in mechanics, it is necessary to consider more general dependence in the complex parameter λ. A particular interesting case is a quadratic family of operators: $L(\lambda)=\lambda^2L_2+\lambda L_1+L_0$, where L_2, L_1 and L_0 are linear operators in $\mathcal{E}$. We shall say that $L(\lambda)$ is a quadratic pencil.

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Let us consider the second order differential equation :

$$L_2(\frac{d^2u}{dt^2})+L_1(\frac{du}{dt})+L_0(u)=0. \tag{1.1}$$

Eq. (1.1) is a model in mechanics for small oscillations of a continuum system in the presence of an impedance force [26]. Now by looking for stationary solutions of (1.1), $u(t)=u_0\mathrm{e}^{\lambda t}$, we have the following equation :

$$(\lambda^2 L_2+\lambda L_1+L_0)u_0=0. \tag{1.2}$$

So Eq. (1.2) is a non-linear eigenvalue problem in the spectral parameter $\lambda\in\mathbb{C}$. We say that λ is a non-linear eigenvalue if there exists $u_0\neq 0$ satisfying (1.2).

The operator L_1 represents a damping term as we see in the following simple example. Let us consider the perturbed wave equation (see [5]) :

$$\frac{\partial^2}{\partial t^2}u-\frac{\partial^2}{\partial^2 x}u-2a\frac{\partial}{\partial t}u=0, \tag{1.3}$$

where $t\in\mathbb{R}$ and $x\in\mathbb{T}:=\mathbb{R}/2\pi\mathbb{Z}$. Here we have $L_2=\mathbb{I}_{\mathcal{E}}$ (identity operator), $L_1=-2a$ and $L_0=-\partial^2/\partial x^2$. The damping term $a<0$ is here constant. So we have to solve (1.3) with periodical boundary conditions. The stationary problem is reduced to the equation :

$$\lambda^2+k^2-2a\lambda=0,\ \ k\in\mathbb{Z}.$$

Then we have for $k^2\geq a^2$ the damped solutions of (1.3) :

$$u_k(t,x)=\exp\Big((a+i\sqrt{k^2-a^2})t+ikx\Big).$$

When a is a function of x we have no explicit formula so we need numerical approximations to compute the damping modes. It is the main goal of this work.

Such generalized eigenvalue problems have appeared in a completely different way. The question was to decide if a class of P.D.E with analytic coefficients preserves or not the analyticity property. To be more explicit, let us consider a P.D.E : $Pu=f$. Assume that f is analytic in some open set Ω, is-it true that u is analytic in Ω ? This is true for elliptic operators. For some example, this question can be reduced to the following (see [23] for more details) : does there exist $\lambda\in\mathbb{C},0\neq u\in\mathcal{S}(\mathbb{R})$ such that

$$\left(-\frac{d^2}{dx^2}+(x^2-\lambda)^2\right)u=0\ \ ? \tag{1.4}$$

Existence of non null solutions for (1.2) and (1.4) is a non trivial problem. For (1.4) it was solved in [33] where it is proved that the generalized eigenfunctions span the Hilbert space $L^2(\mathbb{R})$.