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Fast Evaluation of Linear Combinations of Caputo Fractional Derivatives and Its Applications to Multi-Term Time-Fractional Sub-Diffusion Equations

Guanghua Gao* and Qian Yang

College of Science, Nanjing University of Posts and Telecommunications, Nanjing, Jiangsu 210023, China

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Abstract. In the present work, linear combinations of Caputo fractional derivatives are fast evaluated based on the efficient sum-of-exponentials (SOE) approximation for kernels in Caputo fractional derivatives with an absolute error ϵ, which is a further work of the existing results in [13] (Commun. Comput. Phys., 21 (2017), pp. 650–678) and [16] (Commun. Comput. Phys., 22 (2017), pp. 1028–1048). Both the storage needs and computational amount are significantly reduced compared with the direct algorithm. Applications of the proposed fast algorithm are illustrated by solving a second-order multi-term time-fractional sub-diffusion problem. The unconditional stability and convergence of the fast difference scheme are proved. The CPU time is largely reduced while the accuracy is kept, especially for the cases of large temporal level, which is displayed by numerical experiments.

AMS subject classifications: 65M06, 65M12, 65M15

Key words: Fast evaluation, sum-of-exponentials approximation, multi-term fractional derivatives, stability, convergence.

1. Introduction

As is well-known, factional calculus is non-local and its numerical evaluation is quite different from the integer order one. When one aims to get the value of a fractional derivative at the current time, values of the function on all previous levels need be stored and it brings a huge requirement for computer memory. Hence, some efficient and fast evaluations of fractional derivatives are quite necessary. Zeng et al. [1] proposed a fast memory-saving time-stepping method for both fractional integral and derivative operators using the truncated Laguerre-Gauss quadrature to discretize the

*Corresponding author. *Email addresses:* `gaoguanghua1107@163.com` (G. H. Gao), `861724730@qq.com` (Q. Yang)

kernel in fractional operators. Bohaienko [2] designed a fractional derivative approximation procedure by discretizing convolution kernel using the expression of variable separation so that the computational complexity was reduced from linear to logarithmic. A fast algorithm for the Caputo fractional derivative is developed in [3] based on a nonuniform splitting of the time interval and a polynomial approximation of the kernel function. Wu and Huang [4] investigated the Schwarz waveform relaxation (SWR) algorithm and applied it to time-fractional Cable equations, where a throughout asymptotic analysis was made. Two parareal algorithms for time-fractional differential equations were studied based on two local time-integrators in [5]. A diagonalization-based parallel algorithm was proposed in [6] for computation of time-periodic diffusion equations with fractional Laplacian and the multigrid method was applied as inner solver. A spectral deferred correction algorithm was employed to solve the fractional ordinary differential equations so that the convergence rate was improved in [7]. For solving block triangle Toeplitz-like systems arising from time-fractional partial differential equations (PDEs), the divide-and-conquer strategy and fast approximate inversion method together with the fast Fourier transform were presented in [8] and [9], respectively. By preconditioning the Toeplitz-like linear system coming from fractional diffusion equations, a fast algorithm was studied in [10]. Some preconditioning techniques for matrices arising from discretization of time-space fractional Caputo-Riesz diffusion equations were introduced in [11], where a separable preconditioner was proposed. Li and Chen [12] reviewed some fast algorithms for solving time-fractional PDEs.

It is worth to mention that Jiang et al. [13] considered a fast evaluation of the Caputo fractional derivative based on the sum-of-exponentials (SOE) approximation for kernels, where the acceleration of the $L1$ formula to approximate the Caputo fractional derivative is investigated. A fast evaluation scheme with the finite element approximation in space for fractional sub-diffusion equations was discussed in [14]. As well, in [15], a finite difference scheme for a linearized time fractional KDV equation on unbounded domains was also accelerated using the algorithm presented in [13]. Yan et al. [16] made some further studies on the fast evaluation of Caputo fractional derivatives based on the work in [13], where the acceleration of the higher-order $L2-1_\sigma$ formula derived in [17] was discussed in detail. As an extension of these results, in our current work, linear combinations of Caputo fractional derivatives will be fast evaluated based on the previous ideas in [13] and [16]. We will start from the interpolation approximation for linear combinations of Caputo fractional derivatives proposed in [18] and try to accelerate the evaluation based on an SOE approximation for kernels of each Caputo fractional derivative in linear combinations. Then the numerical accuracy of the presented algorithm will be analyzed and its application to solve a second-order multi-term time-fractional sub-diffusion problem will be illustrated. The stability and convergence of the resultant fast difference scheme will be proved, where one of the key steps in proof is achieved using some more concise techniques than those in [13] and [16].

The outline of this article is as follows. In Section 2, some preliminaries are re-