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High-Order Local Absorbing Boundary Conditions for Fractional Evolution Equations on Unbounded Strips

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Abstract. The study of this paper is two-fold. On the one hand, we reduce the subdiffusion ($0<\alpha<1$) and diffusion-wave ($1<\alpha<2$) problems on unbounded strips to initial boundary value problems (IBVPs) by deriving high-order local artificial boundary conditions (ABCs). After that, the IBVPs with our high-order local ABCs are proved to be stable in the L2-norm. On the other hand, unconditionally stable schemes are constructed to numerically solve the IBVPs by using L1 approximation to discretize the temporal derivative and using finite difference methods to discretize the spatial derivative. We provide the complete error estimates for the subdiffusion case and sketch the proof for the diffusion-wave case. To further reduce the computational and storage cost for the evaluation of the fractional derivatives, the fast algorithm presented in [14] is employed for the case of $0<\alpha<1$ and a similar algorithm for the case of $1<\alpha<2$ is first introduced in this article. Numerical examples are provided to verify the effectiveness and performance of our ABCs and numerical methods.

AMS subject classifications: 35R11, 34A08, 65D20, 65N06, 65N12, 65N15

Key words: Subdiffusion equations, diffsusion-wave equations, anomalous diffusion, artificial boundary methods, fast algorithms, high-order local absorbing boundary conditions.

1 Introduction

The anomalous diffusion has been observed in various complex systems, such as polymers, biopolymers, organisms, liquid crystals, fractals and percolation clusters, and ecosystems [15, 20, 22, 24, 25]. Those anomalous diffusion behaviors can be effectively

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characterized by using fractional differential equations classified as space fractional equations, time fractional equations and time-space fractional equations. In this paper, we consider the numerical solutions of the following anomalous diffusion equations in two dimensional unbounded domains:

$$\begin{cases} {}_0^C D_t^\alpha u(x,y,t) = \Delta u(x,y,t) + f(x,y,t), & (x,y,t) \in \Omega \times (0,T], \quad 0<\alpha<1, \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega, \\ u(x,y,t)|_{(x,y)\in\partial\Omega} = 0, & t \in (0,T], \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}_0^C D_t^\alpha u(x,y,t) = \Delta u(x,y,t) + f(x,y,t), & (x,y,t) \in \Omega \times (0,T], \quad 1<\alpha<2, \\ u(x,y,0) = u_0(x,y), & (x,y) \in \Omega, \\ u_t(x,y,0) = \varphi(x,y), & (x,y) \in \Omega, \\ u(x,y,t)|_{(x,y)\in\partial\Omega} = 0, & t \in (0,T], \end{cases} \tag{1.2}$$

where Ω is an unbounded strip defined as

$$\Omega = \{(x,y) | -\infty < x < 0,\ 0 < y < a,\ 0 \le x < \infty,\ 0 < y < b\},$$

with given constants $b \ge a > 0$. ${}_0^C D_t^\alpha$ represents the Caputo fractional derivative of order α:

$${}_0^C D_t^\alpha u(x,y,t) := \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{1}{(t-s)^\alpha} \frac{\partial u(x,y,s)}{\partial s} ds, & 0<\alpha<1, \\ \dfrac{1}{\Gamma(2-\alpha)} \displaystyle\int_0^t \frac{1}{(t-s)^{\alpha-1}} \frac{\partial^2 u(x,y,s)}{\partial s^2} ds, & 1<\alpha<2. \end{cases}$$

The source term $f(x,y,t)$ and the initial values $u_0(x,y)$, $\varphi(x,y)$ are compactly supported functions.

The numerical computation of the problems (1.1) and (1.2) has two difficulties:

1. the unboundedness of the spatial domain;
2. the expensive computational cost and the large storage resulting from approximating Caputo fractional derivatives.

A common practice to overcome the difficulty of unboundedness is to use artificial boundary methods (ABMs), [7, 11, 12, 29]. The main idea of ABM is introducing artificial boundaries to limit a bounded computational domain of interest, and then designing suitable ABCs to eliminate all the incident waves or flows so that few reflected waves or flows may propagate back into the computational domain. With the constrained ABCs on artificial boundaries, the problems on unbounded domains are reduced to problems on bounded computational domains. Generally, ideal ABCs render the solution of the reduced problem the same as that of the original problem on the unbounded domain. This kind of ABCs is called exact ABCs, and the other is called approximate ABCs.