A TWO-GRID METHOD FOR THE C^0 INTERIOR PENALTY DISCRETIZATION OF THE MONGE-AMPÈRE EQUATION[*]

Gerard Awanou

Department of Mathematics, Statistics, and Computer Science, M/C 249.
University of Illinois at Chicago, Chicago, IL 60607, USA
Email: awanou@uic.edu

Hengguang Li

Department of Mathematics, Wayne State University, 656 W. Kirby, Detroit, MI 48202, USA
Email: hli@math.wayne.edu

Eric Malitz

Department of Mathematics, DePaul University, Chicago, IL 60614, USA
Email: EMALITZ@depaul.edu

Abstract

The purpose of this paper is to analyze an efficient method for the solution of the nonlinear system resulting from the discretization of the elliptic Monge-Ampère equation by a C^0 interior penalty method with Lagrange finite elements. We consider the two-grid method for nonlinear equations which consists in solving the discrete nonlinear system on a coarse mesh and using that solution as initial guess for one iteration of Newton's method on a finer mesh. Thus both steps are inexpensive. We give quasi-optimal $W^{1,\infty}$ error estimates for the discretization and estimate the difference between the interior penalty solution and the two-grid numerical solution. Numerical experiments confirm the computational efficiency of the approach compared to Newton's method on the fine mesh.

Mathematics subject classification: 65N30, 65N55, 35J96.
Key words: Two-grid discretization, Interior penalty method, Finite element, Monge-Ampère.

1. Introduction

In this paper, we prove the convergence of a two grid method for solving the nonlinear system resulting from the discretization of the elliptic Monge-Ampère equation

$$\det(D^2u) = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega, \tag{1.1}$$

with a version of the C^0 interior penalty discretization proposed in [5]. The domain Ω is assumed to be a convex polygonal domain of $\mathbb{R}^2$ and (1.1) is assumed to have a strictly convex smooth solution $u \in C^{k+1}(\overline{\Omega})$ for an integer $k \geq 3$. The function $f \in C^{k-1}(\overline{\Omega})$ is given and satisfies $f \geq c_0$ for a constant $c_0 > 0$ and the function $g \in C(\partial\Omega)$ is also given and assumed to extend to a $C^{k+1}(\overline{\Omega})$ function G. In (1.1), $D^2u = \left(\partial^2u/(\partial x_i\partial x_j)\right)_{i,j=1,2}$ is the Hessian matrix of u and det denotes the determinant operator. Let $V_h \subset H^1(\Omega)$ denote the Lagrange finite element space of degree $k \geq 3$. Let Dv denote the gradient of the function v. Recall that

cof $D^2 v$ denotes the matrix of cofactors of $D^2 v$. The C^0 interior penalty discretization can be written in abstract form as: find $u_h \in V_h$ such that $u_h = g_h$ on $\partial\Omega$ and

$$A(u_h, \phi) = 0, \quad \forall \phi \in V_h \cap H_0^1(\Omega).$$

Here g_h denotes the canonical interpolant in V_h of a continuous extension of g and A is defined in (3.1) below. The discretization has the property that if we denote by $A'(u; v, \phi)$ the Fréchet derivative evaluated at u of the mapping $v \to A(v, \phi)$, then

$$A'(u; v, \phi) = \int_\Omega \left((\operatorname{cof} D^2 u) Dv \right) \cdot D\phi \, dx, \tag{1.2}$$

which gives the weak form of a standard linear elliptic operator. We exploit this property to give quasi-optimal $W^{1,\infty}$ error estimates, and the convergence of a two-grid numerical scheme for solving the discrete nonlinear system. Numerical experiments confirm the computational efficiency of the two-grid method compared to Newton's method on the fine mesh. Two-grid methods were initially analyzed in [13] for quasi-linear problems, and (1.1) is a fully nonlinear equation. The numerical results in [11] used a two-grid method.

Monge-Ampère type equations with smooth solutions on polygonal domains appear in many problems of practical interest. For example they appear in the study of von Kármán model for plate buckling [6]. In addition, for meteorological applications for which other differential operators are discretized with a finite element method, it would be advantageous to use a finite element discretization for the Monge-Ampère operator as well. It is known that when Ω is strictly convex with a smooth boundary, and with our smoothness assumptions on f and g, (1.1) has a smooth solution. There are several discretizations for smooth solutions of (1.1). Provably convergent schemes for non smooth solutions can be used for smooth solutions as well. However the latter have a low order of approximation for smooth solutions. We refer to [8] for example for a review. Because the interior penalty term involves the cofactor matrix of the Hessian, it is very likely that the method proposed in [5] is suitable only for smooth solutions. It does not seem possible to put it in the framework of approximation by smooth solutions proposed in [2], where the right hand side of (1.1) is viewed as a measure.

There has been no previous study of multilevel methods for finite element discretizations of (1.1). A key tool in the proof of convergence of the two-grid method is a $W^{1,\infty}$ norm error estimate for $k \geq 3$. Such estimates were obtained in [10] for quadratic and higher order elements on a smooth domain. But the proof therein relies on an elliptic regularity property of the linearized problem [10, (2.21)]. Unless the domain is a rectangle, we do not expect such an elliptic regularity property to hold for general polygonal domains considered in this paper.

With the quasi-optimal $W^{1,\infty}$ error estimates we obtain a new proof of the optimal H^1 estimates obtained in [5]. Although these estimates are not new, we include netherthless the proof since its ideas are also used in the proof of the convergence of the two-grid method. The version of the C^0 interior penalty discretization proposed in [5] we consider, consists in imposing the boundary condition through interpolation, instead of weakly with a penalty term. In this context, as expected, the proof of the H^1 estimates is simpler than the one given in [5]. In particular, no mesh-dependent norms are used.

The two-grid method consists in solving (1.1) on a coarse mesh of size H and using that solution as initial guess for one iteration of Newton's method on a finer mesh of size h with $H = h^\lambda, 0 < \lambda < 1$. Thus both steps are inexpensive. We prove that the convergence rate does