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Review Article

General Tikhonov Regularization with Applications in Geoscience

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Abstract. The article is devoted to a review of the following new elements of the modern theory of solving inverse problems: (a) general theory of Tikhonov's regularization with practical examples is considered; (b) an overview of a-priori and a-posteriori error estimates for solutions of ill-posed problems is presented as well as a general scheme of a-posteriori error estimation; (c) a-posteriori error estimates for linear inverse problems and its finite-dimensional approximation are considered in detail together with practical a-posteriori error estimate algorithms; (d) optimality in order for the error estimator and extra-optimal regularizing algorithms are also discussed. In addition, the article contains applications of these theoretical results to solving two practical geophysical problems. First, for inverse problems of computer microtomography in microstructure analysis of shales, numerical experiments demonstrate that the use of functions with bounded VH-variation for a piecewise uniform regularization has a theoretical and practical advantage over methods using BV-variation. For these problems, a new algorithm of a-posteriori error estimation makes it possible to calculate the error of the solution in the form of a number. Second, in geophysical prospecting, Tikhonov's regularization is very effective in magnetic parameters inversion method with full tensor gradient data. In particular, the regularization algorithms allow to compare different models in this method and choose the best one, MGT-model.

AMS subject classifications: 65J22, 65K10, 65R30, 65R32

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1 Introduction: Inverse problems in geoscience

Geophysics is one of the sciences in which mathematical methods have been used for more than 100 years. It was in the study of geophysical problems that many mathematical models were developed, which were successfully applied not only in Earth sciences, but also in other applications [48, 56, 79, 82, 83, 87]. The development of many geophysical research methods led to the creation of the theory of inverse problems and then to the methods and algorithms for solving these problems, which are mostly ill-posed. It is for solving inverse problems in Earth sciences that the method of regularization of A. N. Tikhonov was created in the 60s of the XX century [67, 68]. The theory of Tikhonov regularization marked the beginning of the fruitful development of stable methods for solving inverse ill-posed problems. These methods are successfully developed and adapted to the solution of inverse problems in geophysics and other branches of science to the present. In order to understand this, it is enough to indicate that about 100 monographs on this subject have already been published. Among them, for example, are the following [4, 6, 9, 13, 14, 17, 20, 22, 30, 44, 60, 62, 69–72, 79, 89] and many others.

Hundreds of new geophysical models have been established during past decades to be applied for solving inverse problems. The model-based inversion in solid geophysics and atmospheric science has been well understood, as well as the model-based inverse problems for land surface and data-based inverse problems that received much attention from scientists only in recent years (see, e.g. [38, 51, 58, 59, 85–87]). The solution of inverse ill-posed problems is impossible without taking into account a-priori information about the desired solution. For example, in solid geophysics, using gravimetric, magnetic, electromagnetic and seismic data to dig out anomalies underground is always severely ill-posed due to very limited data acquisition during scanning the deep earth [81, 87, 90]. Therefore, integrating different geophysical exploration methods is a tendency. At the same time, the appearance of hyperspectral and multiangular remote sensor enhanced the exploration means, and provided us more spectral and spatial dimension information than before [80, 81]. For this reason, the development and improvement of existing methods for solving ill-posed inverse problems is required, so as to take into account more and more detailed a-priori information about solutions. However, stable solving inverse problems is only part of the processing of geophysical information. As a rule, it is also required to estimate the error of solutions obtained. It is desirable to have this estimate as a number, rather than an asymptotic formula. This requires solving rather complicated math questions for ill-posed problems.

In this article we try to give an overview of two directions in the study of methods for solving ill-posed problems (Section 2). First, we describe modern regularization methods, and in a fairly general form, suitable for solving complex inverse problems (Section 2.1). Secondly, in consideration of error estimation for solutions (Section 2.2), we want to point out the complexity of such a task and show constructive ways to solve it (Sections 2.3, 2.4). We will also present a new error estimation algorithm that allows us to obtain practical estimates (Section 2.5). The important notion of an extra-optimal regularizing