J. Math. Study
doi: 10.4208/jms.v53n2.20.01

Vol. **53**, No. 2, pp. 125-142
June 2020

(Semi-)Nonrelativisitic Limit of the Nonlinear Dirac Equations

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Received Febuary 5, 2019; Accepted August 14, 2019;
published online April 20, 2020.

Dedicated to Professor Jie Shen on the Occasion of his 60th Birthday

Abstract. We consider the nonlinear Dirac equation (NLD) with time dependent external electro-magnetic potentials, involving a dimensionless parameter $\varepsilon\in(0,1]$ which is inversely proportional to the speed of light. In the nonrelativistic limit regime $\varepsilon\ll 1$ (speed of light tends to infinity), we decompose the solution into the eigenspaces associated with the 'free Dirac operator' and construct an approximation to the NLD with $O(\varepsilon^2)$ error. The NLD converges (with a phase factor) to a coupled nonlinear Schrödinger system (NLS) with external electric potential in the nonrelativistic limit as $\varepsilon\to 0^+$, and the error of the NLS approximation is first order $O(\varepsilon)$. The constructed $O(\varepsilon^2)$ approximation is well-suited for numerical purposes.

AMS subject classifications: 35Q41, 35Q55, 81Q05

Key words: Nonlinear Dirac equation, nonrelativistic limit, error estimates.

1 Introduction

In this paper, we consider the nonlinear Dirac equation (NLD) [3, 6, 14, 15, 23, 31] in the following dimensionless form:

$$\begin{cases} i\partial_t\psi(t,\mathbf{x}) = \left[-\frac{i}{\varepsilon}\sum_{j=1}^{3}\alpha_j\partial_j + \frac{1}{\varepsilon^2}\beta + V(t,\mathbf{x})I_4 - \sum_{j=1}^{3}A_j(t,\mathbf{x})\alpha_j + F(\psi)\right]\psi(t,\mathbf{x}), \\ \psi(t=0,\mathbf{x}) = \psi_I^\varepsilon(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

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where $i=\sqrt{-1}$, t is time, $\mathbf{x}=(x_1,x_2,x_3)^T\in\mathbb{R}^3$ is the spatial coordinate vector, $\partial_k=\frac{\partial}{\partial x_k}$ ($k=$ 1,2,3), $\psi:=\psi(t,\mathbf{x})=(\psi_1(t,\mathbf{x}),\psi_2(t,\mathbf{x}),\psi_3(t,\mathbf{x}),\psi_4(t,\mathbf{x}))^T\in\mathbb{C}^4$ is the complex-valued vector wave function of the "spinorfield", $V(t,\mathbf{x})$ and $\mathbf{A}=(A_1(t,\mathbf{x}),A_2(t,\mathbf{x}),A_3(t,\mathbf{x}))^T$ are the real-valued external electric potential and magnetic potential, respectively, $\varepsilon\in(0,1]$ is a dimensionless parameter inversely proportional to the speed of light. $F(\psi)\in\mathbb{C}^{4\times 4}$ is the matrix nonlinearity and one common choice is of the following form

$$F(\psi)=\lambda(\psi^*\beta\psi)\beta+\gamma|\psi|^2 I_4,\quad \lambda,\gamma\in\mathbb{R}, \tag{1.2}$$

where $\psi^*=\overline{\psi^T}$ denotes the conjugate transpose of ψ. The $\lambda\neq 0,\gamma=0$ case is motivated from the famous Soler model [32], for which the solitary solutions and their dynamics have been widely studied in the literature [2,24,34]; The $\lambda=0,\gamma\neq 0$ case is motivated from the Bose-Einstein condensates with a chiral confinement and /or spin-orbit coupling [12, 22,29]. The 4×4 matrices α_1, α_2, α_3 and β are defined as

$$\alpha_1=\begin{pmatrix}\mathbf{0} & \sigma_1\\ \sigma_1 & \mathbf{0}\end{pmatrix},\qquad \alpha_2=\begin{pmatrix}\mathbf{0} & \sigma_2\\ \sigma_2 & \mathbf{0}\end{pmatrix}, \tag{1.3a}$$

$$\alpha_3=\begin{pmatrix}\mathbf{0} & \sigma_3\\ \sigma_3 & \mathbf{0}\end{pmatrix},\qquad \beta=\begin{pmatrix}I_2 & \mathbf{0}\\ \mathbf{0} & -I_2\end{pmatrix}, \tag{1.3b}$$

where σ_1, σ_2, σ_3 are the Pauli matrices given by

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{1.4}$$

The Dirac equation (1.1) conserves the total mass

$$\|\psi(t,\cdot)\|^2:=\int_{\mathbb{R}^3}|\psi(t,\mathbf{x})|^2 d\mathbf{x}=\int_{\mathbb{R}^3}\sum_{j=1}^{4}|\psi_j(t,\mathbf{x})|^2 d\mathbf{x}\equiv\|\psi(0,\cdot)\|^2=\|\psi_I^\varepsilon\|^2,\qquad t\geq 0. \tag{1.5}$$

In addition, if the external electromagnetic potentials are time independent, i.e., $V(t,\mathbf{x})=V(\mathbf{x})$ and $A_j(t,\mathbf{x})=A_j(\mathbf{x})$ ($j=1,2,3$), the NLD (1.1) conserves the energy

$$\begin{aligned}E(t):=&\int_{\mathbb{R}^3}\left(-\frac{i}{\varepsilon}\sum_{j=1}^{3}\psi^*\alpha_j\partial_j\psi+\frac{1}{\varepsilon^2}\psi^*\beta\psi+V(\mathbf{x})|\psi|^2-\sum_{j=1}^{3}A_j(\mathbf{x})\psi^*\alpha_j\psi+\frac{\lambda}{2}|\psi^*\beta\psi|^2+\frac{\gamma}{2}|\psi|^4\right)d\mathbf{x}\\ \equiv& E(0),\quad t\geq 0.\end{aligned} \tag{1.6}$$

There have been many studies on the Dirac equations [1,8,13,16–19,21] including the well-posedness, dynamics of wave packets, etc. The purpose of this paper is to analyze the nonrelativistic limit of the nonlinear Dirac equation (1.1), when $\varepsilon\to 0^+$.

For the linear case, the nonrelativistic limit has been investigated thoroughly in [7, 11,20,23,25,28,30,31,33]. It has been shown that the Dirac equation is a perturbation