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High-Accuracy Numerical Approximations to Several Singularly Perturbed Problems and Singular Integral Equations by Enriched Spectral Galerkin Methods

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Dedicated to Professor Jie Shen on the Occasion of his 60th Birthday

Abstract. Usual spectral methods are not effective for singularly perturbed problems and singular integral equations due to the boundary layer functions or weakly singular solutions. To overcome this difficulty, the enriched spectral-Galerkin methods (ESG) are applied to deal with a class of singularly perturbed problems and singular integral equations for which the form of leading singular solutions can be determined. In particular, for easily understanding the technique of ESG, the detail of the process are provided in solving singularly perturbed problems. Ample numerical examples verify the efficiency and accuracy of the enriched spectral Galerkin methods.

AMS subject classifications: 65N35, 65R20, 41A30

Key words: Singularly perturbed problems, weakly singular integral equations, boundary layers, enriched spectral Galerkin methods, Jacobi polynomials.

1 Introduction

As we know that the numerical error of the most existed numerical methods strictly rely on the regularity of the solution u of the underlying problem. Especially, for the problems with high regularity solutions, spectral methods are capable of providing highly accurate solutions with significantly less unknowns than using a finite-element or finite difference methods [3, 5, 13, 26]. However, usual spectral methods based on orthogonal polynomials/functions do not have satisfactory convergence rate for singularly perturbed

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problems and singular integral equations due to solutions of those problems usually exhibited boundary layer phenomena or singular behaviours. More precisely,

- *singularly perturbed equations* [7,11,15,17,23,28,34]: Given a tiny perturbed parameters ε. The solutions of singularly perturbed equations usually involve the boundary layer, i.e., the solutions change sharply (not tempered) in a narrow domain, such as $e^{-\frac{x}{\varepsilon}}$, so that the usual spectral method cannot catch the information of the boundary layer functions accurately.

- *Singular integral equations* [4,6,10,16,20,24,32,33]: The solutions of many integral equations with weakly singular kernel behave as a summation of Müntz polynomials $\sum_{i=0}^{\infty} x^{r_i}$, $r_i > 0$ (see [4,6]). Usually the index of the leading singular term r_0 is a small positive real number, so spectral methods are inefficient for singular integral equations due to the limited regularity of the solutions.

The results of the spectral approximation to boundary layer functions behaving as $e^{-\frac{x}{\varepsilon}}$ can be considerably improved by using special mapped polynomials [18,19,27,31], where singular mappings are used to establish spectral method with improved algebraic rates of convergence. However, these approximation results are still not uniform in ε. Combing the parameter ε, Schwab and Suri [23] use two-element spectral method (or p version on two elements) to derive a robust exponential rate for boundary layer functions. Different from the singularly perturbed equations, the weak singularity of the solutions of the singular integral equations cannot derived the exponential convergence rate just by two elements due to the derivative of the solution are unbounded. So Wang [32] *et al.* and Yi [33] *et al.* use h-p finite element methods, based on the geometric mesh [14], to handle Volterra integro-differential equations with smooth and weakly singular kernels. Meanwhile, by using a special mapping, Hou *et al.* [16] derive an Müntz spectral method to enhance the convergence rate of the usual spectral method for singular integral equations. In this paper, we adopt the enriched spectral Galerkin method [9] to deal with several singularly perturbed problems and singular integral equations with a few boundary layer functions and leading singular terms which can be determined. The main merit of this method is that the enriched spectral Galerkin method keeps the structure of the usual spectral method. Especially in ESG-II, via a special property of the spectral method, we can obtain an improved numerical approximation just by repeating usual spectral method several times.

The remainder of this paper is organized as follows. In the next section, we provide a general framework of the enriched spectral-Galerkin methods. Moreover, we introduce the classical Jacobi polynomials and their basic properties which will be extensively used in subsequent sections. In Section 3, based on a Jacobi spectral Galerkin scheme and the analysis of the boundary layer functions, we apply ESG-II into several singularly perturbed problems. In Section 4, we study an singular integral equation, derive the form of the singular solutions and then apply ESG-II to obtain accurate solutions. Some concluding remarks are given in Section 5.