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On the Generalized Porous Medium Equation in Fourier-Besov Spaces

Weiliang Xiao[1,*] and Xuhuan Zhou[2]

[1] *School of Applied Mathematics, Nanjing University of Finance and Economics, Nanjing 210023, China;*
[2] *Department of Information Technology, Nanjing Forest Police College, Nanjing 210023, China.*

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Abstract. We study a kind of generalized porous medium equation with fractional Laplacian and abstract pressure term. For a large class of equations corresponding to the form: $u_t+\nu\Lambda^\beta u=\nabla\cdot(u\nabla Pu)$, we get their local well-posedness in Fourier-Besov spaces for large initial data. If the initial data is small, then the solution becomes global. Furthermore, we prove a blowup criterion for the solutions.

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Key words: Porous medium equation, well-posedness, blowup criterion, Fourier-Besov spaces.

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1 Introduction

In this paper, we study the nonlinear nonlocal equation in $\mathbb{R}^n$ of the form

$$\begin{cases} u_t+\nu\Lambda^\alpha u=\nabla\cdot(u\nabla Pu); \\ u(0,x)=u_0. \end{cases} \tag{1.1}$$

Usually, $u=u(t,x)$ is a real-valued function, represents a density or concentration. The dissipative coefficient $\nu>0$ corresponds to the viscous case, while $\nu=0$ corresponds to the inviscid case. In this paper we study the viscid case and take $\nu=1$ for simplicity. The fractional operator Λ^α is defined by Fourier transform as $(\Lambda^\alpha u)^\wedge=|\xi|^\alpha\hat{u}$. P is an abstact operator.

Equation (1.1) here comes from the same proceeding with that of the fractional porous medium equation (FPME) introduced by Caffarelli and Vázquez [5]. In fact, equation

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*Corresponding author. *Email addresses:* `xwltc123@163.com` (W. Xiao), `zhouxuhuan@163.com` (X. Zhou)

(1.1) comes into being by adding the fractional dissipative term $\nu\Lambda^\alpha u$ to the continuity equation $u_t+\nabla\cdot(uV)=0$, where the velocity $V=-\nabla p$ and the velocity potential or pressure p is related to u by an abstract operator $p=Pu$.

The absrtact form pressure term Pu gives a good suitability in many cases. The simplest case comes from a model in groundwater in filtration [1, 20]: $u_t=\triangle u^2$, that is: $\nu=0, Pu=u$. A more general case appears in the fractional porous medium equation [5] when $\nu=0$ and $Pu=\Lambda^{-2s}u, 0<s<1$. In the critical case when $s=1$, it is the mean field equation first studied by Lin and Zhang [16]. Studies on the well-posedness and regularity on those equations we refer to [4, 6, 7, 18, 19, 21, 24] and the references therein.

In the FPME, the pressure can also be represented by Riesz potential as $Pu=\Lambda^{-2s}u=\mathcal{K}*u$, with kernel $\mathcal{K}=c_{n,s}|y|^{2s-n}$. Replacing the kernel $\mathcal{K}$ by other functions in this form: $Pu=\mathcal{K}*u$, equation (1.1) also appears in granular flow and biological swarming, named aggregation equation. The typical kernels are the Newton potential $|x|^\gamma$ and the exponent potential $-e^{-|x|}$.

One of concerned problems on this equation is the singularity of the potential Pu which holds the well-posedness or leads to the blowup solution. Bertozzi and Carrillo [3] show that smooth kernels at origin $x=0$ lead to the global in time solution, meanwhile Li and Rodrigo [15] prove that nonsmoooth kernels lead to blowup phenomenon. Li and Rodrigo [14] studied the well-posedness and blowup criterion of equation (1.1) with the pressure $Pu=\mathcal{K}*u$, where $\mathcal{K}(x)=e^{-|x|}$ in Sobolev spaces. Wu and Zhang [22] generalize their work to require $\nabla\mathcal{K}\in W^{1,1}$ which includes the case $\mathcal{K}(x)=e^{-|x|}$. They take advantage of the controllability in Besov spaces of the convolution $\mathcal{K}*u$ under this condition, as well as the controllability of its gradient $\nabla\mathcal{K}*u$.

In this article we study the well-posedness and blowup criterion of equation (1.1) in Fourier-Besov spaces under an abstract pressure condition

$$\|\widehat{\triangle_k(\nabla Pu)}\|_{L^p}\leq C2^{k\sigma}\|\widehat{\triangle_k u}\|_{L^p}. \tag{1.2}$$

In Fourier-Besov spaces, it is the localization express of the norm estimate

$$\|\nabla Pu\|_{F\dot{B}^s_{p,q}}\leq C\|u\|_{F\dot{B}^{s+\sigma}_{p,q}}. \tag{1.3}$$

Corresponding to the FPME, i.e. $Pu=\Lambda^{-2s}u$, we get $\sigma=1-2s$ obviously. And if $Pu=\mathcal{K}*u$, $\mathcal{K}\in W^{1,1}$ in the aggregation equation, we get $\sigma=1$ when $\mathcal{K}\in L^1$ and $\sigma=0$ when $\nabla\mathcal{K}\in L^1$.

The Fourier-Besov spaces we use here come from Konieczny and Yoneda [12] when deal with the Navier-Stokes equation (NSE) with Coriolis force. Besides, Fourier-Besov spaces have been widely used to study the well-posedness, singularity, self-similar solution, etc. of Fluid Dynamics in various of forms. For instance, the early pseudomeasure spaces PM^α in which Cannone and Karch studied the smooth and singular properties of Navier-Stokes equations [8]. The Lei-Lin spaces $\mathcal{X}^\sigma$ deal with global solutions to the NSE [13] and to the quasi-geostrophic equations (QGE) [2]. The Fourier-Herz spaces $\mathcal{B}^\sigma_q$ in the Keller-Segel system [9], in the NSE with Coriolis force [10] and in the magneto-hydrodynamic equations (MHD) [17].