Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2019-0117

Vol. **28**, No. 2, pp. 768-802
August 2020

A Hodge Decomposition Method for Dynamic Ginzburg–Landau Equations in Nonsmooth Domains — A Second Approach

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Received 10 July 2019; Accepted (in revised version) 3 December 2019

Abstract. In a general polygonal domain, possibly nonconvex and multi-connected (with holes), the time-dependent Ginzburg–Landau equation is reformulated into a new system of equations. The magnetic field $B := \nabla\times\mathbf{A}$ is introduced as an unknown solution in the new system, while the magnetic potential $\mathbf{A}$ is solved implicitly through its Hodge decomposition into divergence-free part, curl-free and harmonic parts, separately. Global well-posedness of the new system and its equivalence to the original problem are proved. A linearized and decoupled Galerkin finite element method is proposed for solving the new system. The convergence of numerical solutions is proved based on a compactness argument by utilizing the maximal L^p-regularity of the discretized equations. Compared with the Hodge decomposition method proposed in [27], the new method has the advantage of approximating the magnetic field B directly and converging for initial conditions that are incompatible with the external magnetic field. Several numerical examples are provided to illustrate the efficiency of the proposed numerical method in both simply connected and multi-connected nonsmooth domains. We observe that even in simply connected domains, the new method is superior to the method in [27] for approximating the magnetic field.

AMS subject classifications: 35Q56, 35K61, 65M12, 65M60

Key words: Superconductivity, reentrant corner, singularity, multi-connected domain, finite element method, convergence, Hodge decomposition.

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1 Introduction

The time-dependent Ginzburg–Landau equation (TDGL) is widely used for numerical simulations of vortex dynamics of superconducting density and magnetic field for type-II superconductors [8,14,19,29]. In this model, the state of a superconductor is described by a complex-valued order parameter ψ, a real vector-valued magnetic potential $\mathbf{A}$, and a real scalar-valued electric potential ϕ. In a two-dimensional domain, the TDGL can be written as (with non-dimensionalisation)

$$\eta\frac{\partial\psi}{\partial t}+\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+(|\psi|^2-1)\psi+i\eta\kappa\psi\phi=0, \tag{1.1}$$

$$\frac{\partial\mathbf{A}}{\partial t}+\nabla\times(\nabla\times\mathbf{A})+\nabla\phi+\mathrm{Re}\left[\overline{\psi}\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]=\nabla\times H, \tag{1.2}$$

with the following notations of curl, divergence and gradient operators:

$$\nabla\times\mathbf{A}=\frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2}, \qquad \nabla\cdot\mathbf{A}=\frac{\partial A_1}{\partial x_1}+\frac{\partial A_2}{\partial x_2},$$
$$\nabla\times H=\left(\frac{\partial H}{\partial x_2},-\frac{\partial H}{\partial x_1}\right), \quad \nabla\psi=\left(\frac{\partial\psi}{\partial x_1},\frac{\partial\psi}{\partial x_2}\right).$$

The time-independent external magnetic field H is given, η and κ are positive physical parameters, and $\overline{\psi}$ denotes the complex conjugate of ψ. The physically interesting quantities in this model are the magnetic field $B=\nabla\times\mathbf{A}$ and the superconductivity density $|\psi|^2$, which satisfies $0\le|\psi|^2\le 1$ and represents the superconducting state of a superconductor. In particular, $|\psi|^2=1$ indicates that the superconductor is in the superconducting state, and $|\psi|^2=0$ indicates the normal state. If the superconductor occupies a domain Ω, then the following physical boundary conditions hold:

$$\left(\frac{i}{\kappa}\nabla\psi+\mathbf{A}\psi\right)\cdot\mathbf{n}=0 \qquad \text{on } \partial\Omega, \tag{1.3}$$

$$\nabla\times\mathbf{A}=H \qquad \text{on } \partial\Omega, \tag{1.4}$$

$$\mathbf{A}\cdot\mathbf{n}=0 \qquad \text{on } \partial\Omega, \tag{1.5}$$

where $\mathbf{n}$ denotes the unit outward normal vector on the boundary $\partial\Omega$.

In addition to (1.1)-(1.2), one needs a gauge condition to determine the solution uniquely [1,12]. For example, the zero electric potential gauge $\phi=0$ and the Lorentz gauge $\phi=-\nabla\cdot\mathbf{A}$ are often used for numerical simulations [10,11,20,23,33,35,36]. The solutions under the different gauges are equivalent in producing the physical quantities $|\psi|^2$ and B (see [12]). In this paper, we focus on the Lorentz gauge $\phi=-\nabla\cdot\mathbf{A}$, which