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EFFICIENT AND ACCURATE CHEBYSHEV DUAL-PETROV-GALERKIN METHODS FOR ODD-ORDER DIFFERENTIAL EQUATIONS*

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Abstract

Efficient and accurate Chebyshev dual-Petrov-Galerkin methods for solving first-order equation, third-order equation, third-order KdV equation and fifth-order Kawahara equation are proposed. Some Sobolev bi-orthogonal basis functions are constructed which lead to the diagonalization of discrete systems. Accordingly, both the exact solutions and the approximate solutions are expanded as an infinite and truncated Fourier-like series, respectively. Numerical experiments illustrate the effectiveness of the suggested approaches.

Mathematics subject classification: 65N35, 33C45, 35J58.
Key words: Chebyshev dual-Petrov-Galerkin method, Sobolev bi-orthogonal polynomials, Odd-order differential equations, Numerical results.

1. Introduction

Spectral methods are based on orthogonal polynomial/function approximations, which possess the high-order accuracy and have gained more and more popularity during the past few decades, see [2, 4, 5, 8, 10, 24, 25] and the references therein. The Fourier trigonometric polynomials e^{ikx}, $k \in \mathbb{Z}$ are the most desirable basis, which are orthogonal with respect to each other under certain Sobolev inner product involving derivatives, thus the corresponding algebraic system is diagonal. This fact together with the availability of the fast Fourier transform (FFT) makes the Fourier spectral method be an ideal approximation approach for differential equations with periodic boundary conditions. If a Fourier method is applied to a non-periodic problem, it inevitably induces the so-called Gibbs phenomenon, and reduces the global convergence rate to $O(N^{-1})$. Consequently, one should not apply a Fourier method to problems with non-periodic boundary conditions. Instead, the Chebyshev spectral methods [4, 11–13, 17] are of our greatest interests due to the FFT for Chebyshev polynomials.

Standard Chebyshev spectral methods have been extensively investigated for solving second-order and fourth-order differential equations (see, e.g., [22]). For the one-dimensional fourth-order linear equation, Shen [22] presented a basis

$$\varphi_k(x) = T_k(x) - \frac{2(k+2)}{k+3}T_{k+2}(x) + \frac{k+1}{k+3}T_{k+4}(x), \qquad 0 \le k \le N-4,$$

with $T_k(x)$ being the kth Chebyshev polynomial. Note that the matrix with the term $(\partial_x^2\varphi_k, \partial_x^2(\varphi_l\omega))$ in the resulting linear system is not sparse but possesses special structure, where $\omega(x)$

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is the Chebyshev weight function. Benefiting from these special matrix structures, Shen [22] further derived some efficient algorithms. However, there is only a limited body of literature on spectral methods for odd-order equations. This is partly due to the fact that: (i) for the classical Chebyshev-Galerkin spectral methods, the discrete systems are not sparse and the condition numbers increase like $O(N^k)$ for the k-order boundary problem, where N is the number of modes; (ii) for the direct collocation methods, the discrete systems are also not sparse and the condition numbers increase like $O(N^{2k})$ for k-order boundary problems, and often exhibit unstable modes if the collocation points are not properly chosen (see, e.g., [14, 21]).

Since the main differential operators in odd-order differential equations are not symmetric, it is reasonable to use the Petrov-Galerkin spectral method. Recently, Ma and Sun [18, 19] developed an efficient Legendre-Petrov-Galerkin and Chebyshev collocation method for the third-order differential equations. By choosing appropriate basis functions, the resulting linear system is sparse. Shen [23] proposed a Legendre dual-Petrov-Galerkin spectral method for the third and higher odd-order equations, and obtained linear systems which are compactly sparse. Moreover, Shen and Wang [26] presented Legendre and Chebyshev dual-Petrov-Galerkin spectral methods for the first-order hyperbolic equations, which are always stable without any restriction on the coefficients, the resulting linear systems are also compactly sparse for problems with constant coefficients and well conditioned for problems with variable coefficients by a preconditioning method.

In this paper, we consider the first and third order differential equations by using Chebyshev dual-Petrov-Galerkin method. As pointed out in [22], it is very important to choose an appropriate basis such that the resulting linear system is as simple as possible. Motivated by the success work in [1, 16, 27], the main purpose of this paper is to construct new Fourier-like Sobolev bi-orthogonal basis functions [7,20], such that the resulting linear systems are diagonal.

The main advantages of the suggested algorithms include:

- the exact solutions and the approximate solutions can be represented as infinite and truncated Fourier-like series, respectively;
- the condition numbers of the resulting algebraic systems are always equal to one;
- the computational cost is much less than that of the classical Chebyshev dual-Petrov-Galerkin method.

The remainder of this paper is organized as follows. In Section 2, we introduce the Chebyshev polynomials and their basic properties. In Section 3, we construct two kinds of Sobolev bi-orthogonal Chebyshev polynomials corresponding to the odd-order differential equations, and propose new Chebyshev dual-Petrov-Galerkin methods. Some numerical results are presented in Section 4 to demonstrate the effectiveness and accuracy.

2. Notations and Preliminaries

Let I be a certain interval and $\omega(x)$ be a weight function in the usual sense. For integer $r \geq 0$, we define the weighted Sobolev spaces $H^r_\omega(I)$ as usual, with the inner product $(u, v)_{r,\omega}$, the semi-norm $|v|_{r,\omega}$ and the norm $\|v\|_{r,\omega}$. We omit the subscript $\omega(x)$ whenever $\omega(x) \equiv 1$.

We first recall the Chebyshev polynomials. Let $I = (-1, 1)$ and $T_k(x)$ be the Chebyshev polynomial of degree k, which is the eigenfunction of the singular Sturm-Liouville problem