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Estimates of Dirichlet Eigenvalues for One-Dimensional Fractal Drums

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Dedicated to Professor Weiyi Su on the occasion of her 80th birthday

Abstract. Let Ω, with finite Lebesgue measure $|\Omega|$, be a non-empty open subset of $\mathbb{R}$, and $\Omega=\bigcup_{j=1}^{\infty}\Omega_j$, where the open sets Ω_j are pairwise disjoint and the boundary $\Gamma=\partial\Omega$ has Minkowski dimension $D\in(0,1)$. In this paper we study the Dirichlet eigenvalues problem on the domain Ω and give the exact second asymptotic term for the eigenvalues, which is related to the Minkowski dimension D. Meanwhile, we give sharp lower bound estimates for Dirichlet eigenvalues for such one-dimensional fractal domains.

Key Words: One-dimensional fractal drum, Dirichlet eigenvalues, Pólya conjecture, Minkowski dimension.

AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

1 Introduction and main results

Let Ω, with boundary $\Gamma=\partial\Omega$ be a non-empty open subset of $\mathbb{R}^n$ $(n\geq 1)$. We assume that Ω has finite Lebesgue measure $|\Omega|$. Then we consider the following Dirichlet eigenvalues problem:

$$\begin{cases} -\Delta u=\lambda u & \text{in } \Omega,\\ u=0 & \text{on } \Gamma. \end{cases} \tag{1.1}$$

As is well-known (or see [4]), the problem (1.1) has a sequence of discrete eigenvalues, which can be ordered, after counting finite multiplicity as

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_k\leq\cdots \quad \text{and} \quad \lambda_k\to+\infty \quad \text{as} \quad k\to+\infty.$$

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In 1911, Weyl studied the problem (1.1) and obtained the following first asymptotic term for a bounded open Ω in $\mathbb{R}^n$ (see [16,17]):

$$\lambda_k \sim \frac{4\pi^2 k^{\frac{2}{n}}}{(B_n|\Omega|)^{\frac{2}{n}}} \quad \text{as} \quad k \to +\infty, \tag{1.2}$$

where B_n is the volume of the unit ball in $\mathbb{R}^n$ and we say that $f_k \sim g_k$ as $k \to +\infty$ means $\frac{f_k}{g_k} \to 1$ as $k \to +\infty$.

In 1961, Pólya [14] gave his conjecture for any bounded open subset Ω in $\mathbb{R}^n$, the Dirichlet eigenvalues has the following lower bounds for any $k \geq 1$,

$$\lambda_k \geq \frac{4\pi^2 k^{\frac{2}{n}}}{(B_n|\Omega|)^{\frac{2}{n}}}, \tag{1.3}$$

also he proved his conjecture (1.3) would be true when Ω is a plane domain which tiles $\mathbb{R}^2$. Later in 1983, Peter Li and Yau [12] proved for general bounded domain Ω with smooth boundary, the Dirichlet eigenvalues had the following lower bounds:

$$\sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2 k^{\frac{2+n}{n}}}{(B_n|\Omega|)^{\frac{2}{n}}} \quad \text{for any} \quad k \geq 1. \tag{1.4}$$

In this direction, there are a lot of research works on eigenvalues for smooth domains (or regular domains), e.g., one can see [7,8,13,15].

However, when Ω is fractal domain, i.e., its boundary Γ is "fractal", the situation will be more complicated (cf. [2,3,5,9–11]). Here in this paper, we shall study the case for Ω is one-dimensional fractal string (see the definition below). We shall give the results for second asymptotic term and precise lower bound estimates of Dirichlet eigenvalues for such kinds of fractal sets.

Let us start to consider the open bounded subset $\Omega \in \mathbb{R}^n$ with boundary Γ, we shall first give some definitions about Minkowski measurability and Minkowski dimension as followes:

Definition 1.1. Let $\Gamma_\varepsilon = \{x : d(x,\Gamma) < \varepsilon\}$, where $d(x,\Gamma)$ denotes the Euclidean distance of x to the boundary Γ. For $d \in [n-1,n]$, the (d-dimensional) upper Minkowski content and lower Minkowski content of Γ are given by

$$\mathcal{M}^*(d;\Gamma) \triangleq \limsup_{\varepsilon\to 0^+} \varepsilon^{-(n-d)}|\Gamma_\varepsilon \cap \Omega|, \quad \mathcal{M}_*(d;\Gamma) \triangleq \liminf_{\varepsilon\to 0^+} \varepsilon^{-(n-d)}|\Gamma_\varepsilon \cap \Omega|.$$

The Minkowski dimension of Γ is defined as

$$D = \inf_{d\in[n-1,n]} \{d : \mathcal{M}^*(d;\Gamma) < +\infty\}. \tag{1.5}$$

If $0 < \mathcal{M}_*(D;\Gamma) = \mathcal{M}^*(D;\Gamma) = \mathcal{M}(D;\Gamma) < +\infty$, then Γ is D-dimensional Minkowski measurable with Minkowski measure $\mathcal{M}(D;\Gamma)$.