Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2020-0051

Vol. **14**, No. 1, pp. 242-260
February 2021

On the Evolutionary Dynamics of the Cahn-Hilliard Equation with Cut-Off Mass Source

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Received 31 March 2020; Accepted (in revised version) 12 July 2020

Abstract. We investigate the effect of cut-off logistic source on evolutionary dynamics of a generalized Cahn-Hilliard (CH) equation in this paper. It is a well-known fact that the maximum principle does not hold for the CH equation. Therefore, a generalized CH equation with logistic source may cause the negative concentration blow-up problem in finite time. To overcome this drawback, we propose the cut-off logistic source such that only the positive value greater than a given critical concentration can grow. We consider the temporal profiles of numerical results in the one-, two-, and three-dimensional spaces to examine the effect of extra mass source. Numerical solutions are obtained using a finite difference multigrid solver. Moreover, we perform numerical tests for tumor growth simulation, which is a typical application of generalized CH equations in biology. We apply the proposed cut-off logistic source term and have good results.

AMS subject classifications: 65M06, 65M55, 65Z05, 68U20

Key words: Cahn-Hilliard equation, logistic source, finite difference method, tumor growth application.

1. Introduction

In this paper we investigate numerically the evolutionary dynamics of the following generalized Cahn-Hilliard (CH) equation with nonlinear source $\gamma(\phi(\mathbf{x},t))$ and further examine the effect of cut-off source $\gamma_c(\phi(\mathbf{x},t))$,

$$\frac{\partial \phi(\mathbf{x},t)}{\partial t} = \Delta \mu(\mathbf{x},t) + \gamma\big(\phi(\mathbf{x},t)\big), \qquad \mathbf{x} \in \Omega, \quad t > 0, \tag{1.1}$$

$$\mu(\mathbf{x},t) = F'\big(\phi(\mathbf{x},t)\big) - \epsilon^2 \Delta \phi(\mathbf{x},t), \tag{1.2}$$

$$\mathbf{n} \cdot \nabla \phi(\mathbf{x},t) = \mathbf{n} \cdot \nabla \mu(\mathbf{x},t) = 0, \qquad \mathbf{x} \in \partial\Omega, \tag{1.3}$$

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where $\phi(\mathbf{x}, t)$ $(0 \leq \phi \leq 1)$ is the order parameter of phase-field model, which represents the concentration of one species in the binary mixtures in the domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$. The variational derivative $F'(\phi) = \phi^3 - 1.5\phi^2 + 0.5\phi$ is derived from $F(\phi) = 0.25\phi^2(\phi - 1)^2$, which is a double-well potential, a positive constant ϵ is related to the thickness of interfacial transition, and $\mathbf{n}$ is the outward normal vector at the domain boundary. Note that Eqs. (1.1) and (1.2) without $\gamma(\phi)$ are the CH equation, which is originally devised to represent the coarsening dynamics and phase separations in binary alloy [5]. We refer the readers to a review paper [21] and the references therein for more detailed physical, mathematical, and numerical derivations for the CH equation. The CH type models are applied in various fields such that bacterial films [20], thin films [28], image inpainting [3], population dynamics [7], etc. The CH equation with Eq. (1.3) yields the conservation of mass of ϕ, the existence of global attractors in finite dimensional space has been proved as the follow-up research [27]. In addition, the equilibrium solutions of the CH equation were studied especially on the effect of the dimensions of space [23].

In slightly different sense, the generalized model, Eqs. (1.1) and (1.2), has been employed throughout biological entities, for instance, the growth simulation of tumors. Typically in tumor growth research, appropriate mathematical models and computational simulations have important roles on diagnosis and treatment, along with clinical and experimental data [29]. Since the interactions between cellular proliferation, adhesion, and other properties are all concerned with tissue growth, it is important to simulate diffuse-interface kinetics in which tissue growth occurs. Therefore, a lot of studies related on tumor growth modeling have been conducted. Huang *et al.* [13] addressed numerical computations of biological models on unbounded domains. Efficient local absorbing boundary conditions and linearized finite difference method were built and presented to solve the reduced problem. The authors in [11, 34] developed a thermodynamically consistent diffuse-interface model of multispecies tumor growth with complex morphologies by using the CH type model with substrate components. They presented the proof of thermodynamic consistency of the multispecies mixture model and provided numerical simulations of unstable avascular tumor growth and malignant development such as tumor invasion, tumor-induced angiogenesis in both two- and three-dimensional spaces. Another related study of thermodynamic consistency is a four-species tumor growth model consisting of both tumor and healthy cells and extracellular species, which is proposed by the authors in [12]. They presented various numerical simulation results with several model parameters, and confirmed that the profit of usage of diffuse-interface models in order to incorporate multiphase in a single model. Jiang *et al.* [16] described a phase-field based diffuse-interface model for cell growth, division, and packing of multi-cell aggregation using finite element method. Vilanova *et al.* [33] presented a model of tumor angiogenesis for vascular regression and regrowth using the Allen-Cahn type equation which is widely used in diffuse-interface modeling. They controlled the growth and regression with the chemical free energy. On the one hand, a level-set based method was used to capture the geometric influences of curvature on the evolution of cell density and tissue shape and