INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 17, Number 6, Pages 794–805

A CONFORMING DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD: PART III

XIU YE AND SHANGYOU ZHANG

Abstract. The conforming discontinuous Galerkin (CDG) finite element methods were introduced in [12] on simplicial meshes and in [13] on polytopal meshes. The CDG method gets its name by combining the features of both conforming finite element method and discontinuous Galerkin (DG) finite element method. The goal of this paper is to continue our efforts on simplifying formulations for the finite element method with discontinuous approximation by constructing new spaces for the gradient approximation. Error estimates of optimal order are established for the corresponding CDG finite element approximation in both a discrete H^1 norm and the L^2 norm. Numerical results are presented to confirm the theory.

Key words. Weak gradient, discontinuous Galerkin, stabilizer/penalty free, finite element methods, second order elliptic problem.

1. Introduction

Finite element methods with discontinuous approximation are flexible in finite element construction and mesh generation. However, when discontinuous approximation is used, finite element formulations tend to be more complex to ensure connection of discontinuous function across element boundary. For example, stabilizing/penalty terms are often needed in finite element methods with discontinuous approximations to enforce connection of discontinuous functions across element boundaries [2, 4, 5, 6, 7, 9, 10]. Removing stabilizing term from discontinuous finite element methods will reduce the complexity of formulation and computer programming.

Aiming on simplifying finite element formulation with discontinuous approximation, conforming discontinuous Galerkin finite element methods have been developed in [12] on simplicial mesh and in [13] on polytopal mesh for the following model problem: seek an unknown function u satisfying

$$-\Delta u = f \quad \text{in } \Omega, \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where Ω is a bounded polytopal domain in $\mathbb{R}^d$. The weak form of the problem (1)-(2) is given as follows: find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega). \tag{3}$$

Conforming discontinuous Galerkin finite element method by name maintains the flexibility of DG methods and the features of conforming finite element method such as simple formulation: find $u_h \in V_h$ such that

$$(\nabla_w u_h, \nabla_w v) = (f, v) \quad \forall v \in V_h, \tag{4}$$

where ∇_w is a approximation of gradient ∇. Construction of the space to approximate ∇ is the key of maintaining ultra simple formulation (4). In [13], gradient is approximated by a polynomial of order $j = k + n - 1$ with n the number of sides

Received by the editors August 18, 2020.
2000 *Mathematics Subject Classification.* 65N15, 65N30, 35J50 .

of polygonal element. This result has been improved in [1] by reducing the degree of polynomial j. In [8], Wachspress coordinates are used to approximate ∇, which are usually rational functions, instead of polynomials.

The goal of this paper is to develop a new CDG finite element method with a different philosophy to approximate gradient ∇. In this method, we use piecewise low order polynomial to approximate ∇_w instead of using one piece high order polynomial in [13]. Optimal order error estimates are established for the corresponding conforming DG approximations in both a discrete H^1 norm and the L^2 norm. Numerical results are presented verifying the theorem.

2. Preliminaries

For any given polygon $D \subseteq \Omega$, we use the standard definition of Sobolev spaces $H^s(D)$ with $s \geq 0$. The associated inner product, norm, and semi-norms in $H^s(D)$ are denoted by $(\cdot,\cdot)_{s,D}$, $\|\cdot\|_{s,D}$, and $|\cdot|_{s,D}$, respectively. When $s = 0$, $H^0(D)$ coincides with the space of square integrable functions $L^2(D)$. In this case, the subscript s is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript D is also suppressed when $D = \Omega$.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [11]. Denote by $\mathcal{E}_h$ the set of all edges/faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \backslash \partial\Omega$ be the set of all interior edges/faces. For simplicity, we will use term edge for edge/face without confusion.

Let $P_k(K)$ consist all the polynomials degree less or equal to k defined on K. A finite element space V_h is defined for $k \geq 1$ as

$$V_h = \left\{ v \in L^2(\Omega) : \ v|_T \in P_k(T), \ \ T \in \mathcal{T}_h \right\}. \tag{5}$$

Let T_1 and T_2 be two polygons/polyhedrons in $\mathcal{T}_h$ sharing $e \in \mathcal{E}_h$. For $e \in \mathcal{E}_h$ and $v \in V_h + H^1(\Omega)$, the jump $[v]$ is defined as

$$[v] = v \quad \text{if } e \subset \partial\Omega, \quad [v] = v|_{T_1} - v|_{T_2} \ \text{ if } e \in \mathcal{E}_h^0. \tag{6}$$

The order of T_1 and T_2 is not essential. For $e \in \mathcal{E}_h$ and $v \in V_h + H^1(\Omega)$, the average $\{v\}$ is defined as

$$\{v\} = 0 \quad \text{if } e \subset \partial\Omega, \quad \{v\} = \frac{1}{2}(v|_{T_1} + v|_{T_2}) \ \text{ if } e \in \mathcal{E}_h^0. \tag{7}$$

The space $H(div;\Omega)$ is defined as the set of vector-valued functions on Ω which, together with their divergence, are square integrable; i.e.,

$$H(div;\Omega) = \left\{ \mathbf{v} \in [L^2(\Omega)]^d : \ \nabla\cdot\mathbf{v} \in L^2(\Omega) \right\}.$$

For any $T \in \mathcal{T}_h$, it can be divided in to a set of disjoint triangles T_i with $T = \cup T_i$. Then $\Lambda_h(T)$ can be defined as

$$\Lambda_k(T) = \{ \mathbf{v} \in H(div, T), \ \mathbf{v}|_{T_i} \in RT_k(T) \}, \tag{8}$$

where $RT_k(T)$ is the usual Raviart-Thomas element of order k [3].

For a function $v \in V_h + H^1(\Omega)$, its weak gradient $\nabla_w v$ is defined as a piecewise vector valued polynomial such that $\nabla_w v|_T \in \Lambda_k(T)$ and satisfies the following equation,

$$(\nabla_w v, \mathbf{q})_T = -(v, \nabla\cdot\mathbf{q})_T + \langle \{v\}, \mathbf{q}\cdot\mathbf{n} \rangle_{\partial T} \qquad \forall \mathbf{q} \in \Lambda_k(T). \tag{9}$$