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Isoperimetric Type Inequalities and Hypersurface Flows

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Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays

Abstract. New types of hypersurface flows have been introduced recently with goals to establish isoperimetric type inequalities in geometry. These flows serve as efficient paths to achieve the optimal solutions to the problems of calculus of variations in geometric setting. The main idea is to use variational structures to develop hypersurface flows which are monotonic for the corresponding curvature integrals (including volume and surface area). These new geometric flows pose interesting but challenging PDE problems. Resolution of these problems have significant geometric implications.

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1 Introduction

It has been observed that the isoperimetric difference is decreasing along the curve shortening flow [9]

$$X_t = -\kappa\nu, \tag{1.1}$$

where κ is the curvature of the boundary and ν the outer normal. Let $|\Omega|$ and $|\partial\Omega|$ be the area and perimeter of a bounded domain $\Omega \subset \mathbb{R}^2$. Along the curve shortening flow, let $\Omega(t)$ be the domain at time t. It follows from the Gauss-Bonnet Theorem and the Cauchy-Schwarz inequality that, the isoperimetric difference

$$\mathcal{D}(\Omega(t)) = |\partial\Omega(t)|^2 - 4\pi|\Omega(t)|$$

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is monotonic decreasing (and strictly decreasing if $\Omega(t)$ is not a round ball). The convergence of curve shorting flow (1.1) yields the classical isoperimetric inequality in $\mathbb{R}^2$.

If $\Omega_0 \subset \mathbb{N}$ is an optimal domain in a space $\mathbb{N}$ of the isoperimetric problem, then $\partial\Omega_0$ is a hypersurface of constant mean curvature. The isoperimetric problem can be considered as a problem of calculus of variation: for given A, find a domain Ω such that $\mathcal{V}(\Omega)=|\Omega|$ is of least volume among all domains in $\mathbb{N}$ with $\mathcal{A}(\Omega)=|\partial\Omega|=A$. We search for an *effective path under volume constraint* to achieve an optimal domain. For any variational vector field η, let $f\nu$ be its normal component. Then

$$\delta_\eta \mathcal{V} = \int_{\partial\Omega} f d\mu_g,$$

where g is the induced metric of the boundary $\partial\Omega$. The volume is preserved if and only if

$$\int_{\partial\Omega} f d\mu_g = 0.$$

That is, the normal component f is orthogonal to the kernel of Δ_g. This is the case if and only if

$$f = \Delta_g \Phi, \tag{1.2}$$

for some Φ on $\partial\Omega$.

One has freedom to pick any Φ. We would like to search Φ such that to ensure the *monotonicity* of the hypersurface area. Since Ω may evolve, we look for Φ which is defined in $\mathbb{N}$ (or a region of $\mathbb{N}$).

Let's first consider $\mathbb{N}=\mathbb{R}^{n+1}$. For any bounded domain $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega$, let X denote the position vector of the boundary surface, and let $|X|$ be the distance from the origin. In (1.2), we choose

$$\Phi = \frac{|X|^2}{2}.$$

Set $u = \langle X, \nu \rangle$ as the support function of $\partial\Omega$,

$$\Delta_g \Phi = \Delta_g \frac{|X|^2}{2} = n - Hu.$$

This yields mean curvature type flow introduced in [15],

$$\partial_t X = (n - Hu)\nu. \tag{1.3}$$

Indeed, function $\Phi = \frac{|X|^2}{2}$ carries some very special geometric properties:

$$\nabla_g^2 \Phi = \nabla_g^2 \frac{|X|^2}{2} = g - uh, \tag{1.4}$$