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The Reflexive Selfadjoint Solutions to Some Operator Equations

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Dedicated to Prof. Roger Penrose for his 90th birthday.

Abstract. In this paper, we study the existence of the reflexive, reflexive selfadjoint and reflexive positive solutions to some operator equations with respect to the generalized reflection operator dual (P,Q). We derive necessary and sufficient conditions for the solvability of these equations and provide a detailed description of the solutions in the solvable case by using the Moore-Penrose inverses.

AMS subject classifications: 15A09, 47A05

Key words: (P,Q) reflexive solution, operator equation, positive operator.

1 Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable, infinite dimensional, complex Hilbert spaces. We denote the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ by $\mathcal{B}(\mathcal{H},\mathcal{K})$ and by $\mathcal{B}(\mathcal{H})$ when $\mathcal{H}=\mathcal{K}$. For $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$, let A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ be the adjoint, the range and the null space of A, respectively. $\overline{\mathcal{R}(A)}$ is the closure of $\mathcal{R}(A)$. An operator $A\in\mathcal{B}(\mathcal{H})$ is said to be injective if $\mathcal{N}(A)=\{0\}$. A is densely defined if the domain of A is a dense subset of $\mathcal{H}$ and the range of A is contained within $\mathcal{H}$. A is said to be positive if $(Ax,x)\geq 0$ for all $x\in\mathcal{H}$. Note that the positive operator

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A has a unique square root $A^{\frac{1}{2}}$. Let $P_{\mathcal{M}}$ be the orthogonal projection on closed subspace $\mathcal{M}\subseteq\mathcal{H}$. $I_{\mathcal{M}}$ denotes the identity onto $\mathcal{M}$ or I if there is no confusion. For $A,B,P,Q\in\mathcal{B}(\mathcal{H})$, denote by

$$\begin{aligned} &|A|=(A^*A)^{\frac{1}{2}}, && [A,B]=AB-BA,\\ &A_P=\begin{pmatrix} A(I+P)\\ A(I-P)\end{pmatrix}, && B_Q=\begin{pmatrix} B(I+Q)\\ B(I-Q)\end{pmatrix}. \end{aligned} \tag{1.1}$$

We say that $P\in\mathcal{B}(\mathcal{H})$ is a reflection operator if $P^*=P$ and $P^2=I$. For two reflection operators P and Q, denote by

$$\begin{aligned} &\mathcal{B}_{RPQ}(\mathcal{H})=\{X\in\mathcal{B}(\mathcal{H}):X=PXQ\},\\ &\mathcal{B}_{APQ}(\mathcal{H})=\{Y\in\mathcal{B}(\mathcal{H}):Y=-PYQ\}. \end{aligned}$$

The operator $X\in\mathcal{B}_{RPQ}(\mathcal{H})$ (resp. $Y\in\mathcal{B}_{APQ}(\mathcal{H})$) is said to be (P,Q) reflexive (resp. (P,Q) anti-reflexive) operator with respect to the reflection operator pair (P,Q) [2,3]. By $\mathcal{B}^{\mathcal{S}}(\mathcal{H})$ and $\mathcal{B}^{+}(\mathcal{H})$ we denote the set of all selfadjoint elements and all positive elements in $\mathcal{B}(\mathcal{H})$, respectively. Denote by

$$\begin{aligned} &\mathcal{B}^{S}_{RP}(\mathcal{H})=\mathcal{B}_{RPP}(\mathcal{H})\cap\mathcal{B}^{\mathcal{S}}(\mathcal{H}),\\ &\mathcal{B}^{+}_{RP}(\mathcal{H})=\mathcal{B}_{RPP}(\mathcal{H})\cap\mathcal{B}^{+}(\mathcal{H}). \end{aligned}$$

The (P,Q) reflexive and anti-reflexive operators have many applications in system and control theory, in engineering, in scientific computations and various other fields [2,3,5,15]. The positive solutions to the equation $AX=C$ were studied in [6–8, 13, 14, 19, 20] for different setting in Hilbert space or Hilbert C^*-module. The equation $XA^*-AX^*=B$ was studied in [1,9].

The purpose of this paper is to provide a new approach to the study of (P,Q) reflexive solution and (P,P) reflexive self-adjoint and (P,P) reflexive positive solution respectively to the operator system $AX=B$. We get the necessary and sufficient conditions for the existence of a solution and obtain the general expression of the solution in the solvable case.

The paper is organized in the following way. In Section 2, we will recall some results about operators on Hilbert space. In Section 3, we will give the necessary and sufficient conditions for the existence of a (P,Q) reflexive solution to the operator equation $AX=B$ and provide a formula for the general solution to this operator equation. In Section 4, we consider the existence and expressions for the (P,Q) reflexive and anti-reflexive solutions to the operator equation $AXB=C$. In Section 5, we apply the obtained results to study the (P,Q) reflexive solution and (P,P) reflexive self-adjoint solution to the operator system $AX=B$ and $XC=D$. A new result concerning the (P,Q) reflexive solution of the operator equation $A^*X+X^*A=B$ is derived in this section.