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An Efficient and Unconditionally Convergent Galerkin Finite Element Method for the Nonlinear Schrödinger Equation in High Dimensions

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Abstract. In this paper, we aim to propose and analyze a linearized three-level Galerkin finite element method (FEM) for the nonlinear Schrödinger equation with a general nonlinearity and an external potential. Compared with the existing results in literature, under a weaker assumption on both the exact solution and the nonlinear term, we give a concise proof to establish the optimal L^2 error estimate without any grid-ratio restriction. Besides the standard energy method, the key tools used in our analysis are an induction argument and several Sobolev inequalities. Numerical results are reported to verify our theoretical analysis.

AMS subject classifications: 65N12, 65N15

Key words: Nonlinear Schrödinger equation, linearized Galerkin FEM, unconditional convergence, optimal error estimate.

1 Introduction

As an important and famous mathematical model derived in quantum dynamics, the nonlinear Schrödinger (NLS) equation is widely used to describe numerous physical phenomena. In this paper, we consider the following generalized nonlinear Schrödinger (GNLS) equation

$$i\psi_t+\Delta\psi+V(\mathbf{x})\psi+g(|\psi|^2)\psi=0,\quad \mathbf{x}\in\Omega,\quad 0<t\leq T, \tag{1.1}$$

with initial condition

$$\psi(\mathbf{x},0)=\psi_0(\mathbf{x}),\quad \mathbf{x}\in\overline{\Omega}, \tag{1.2}$$

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and homogeneous Dirichlet boundary condition

$$\psi(\mathbf{x},t)=0, \quad \mathbf{x}\in\partial\Omega, \tag{1.3}$$

where $i=\sqrt{-1}$ is the imaginary unit, $\psi=\psi(\mathbf{x},t)$ is the complex unknown function defined in $\overline{\Omega}\times[0,T]$, and $\Omega\subset\mathbb{R}^d$ for $d=2$ or 3 is bounded and convex (or smooth), $V=V(\mathbf{x})$ is an external potential function, $g=g(s)$ is a given real-valued nonlinear function from $\mathbb{R}^+$ to $\mathbb{R}$, which has different forms in different physical problems [16], such as polynomial function $g(s)=s^\mu$ $(\mu>0)$, exponential function $g(s)=1-e^{-s}$, logarithm function $f(s)=\ln(1+s)$ and rational function $g(s)=-\frac{4s}{1+s}$. A popular one is a cubic NLS equation, where $V(\mathbf{x})=0$ and $g(s)=\beta s$ with real constant β. Here we consider a more general case and mere assume that $g\in C^1(\mathbb{R})$, while it is assumed that $g\in C^2(\mathbb{R})$ in [13,43,44].

The initial-boundary value problem (1.1)-(1.3) enjoys numerous conservation laws [3, 4], such as the the total mass

$$M(t)=:\int_\Omega|\psi(\mathbf{x},t)|^2d\mathbf{x}\equiv M(0), \quad t\geq 0, \tag{1.4}$$

and energy

$$E(t)=:\int_\Omega\Big(|\nabla\psi(\mathbf{x},t)|^2-V(\mathbf{x})|\psi(\mathbf{x},t)|^2-G\big(|\psi(\mathbf{x},t)|^2\big)\Big)d\mathbf{x}\equiv E(0), \quad t\geq 0, \tag{1.5}$$

where

$$G(\mu)=\int_0^\mu g(s)ds$$

is a primitive function of g.

A large number of numerical methods for the NLS equation have been developed in the literature, such as the time-splitting pseudospectral method [8, 10, 11, 41], finite difference method [16, 18, 22, 32, 40, 45, 49, 50], finite element method [1, 2, 23], discontinuous Galerkin method [27, 51, 52], meshless method [19, 20] and Runge-Kutta or Crank-Nicolson pseudo-spectral method [14, 21, 39]. Comparisons between different numerical methods for the NLS equation are studied in [9, 16, 35, 42] and references therein.

Error estimates for different numerical methods of the NLS equation in one dimension have been established in [6, 7, 26, 33]. For the analysis of splitting error of the time-splitting or split-step method for the NLS equation, we refer to [12,17,34,36,41] and references therein. For the error estimates of the implicit Runge-Kutta finite element method for NLS, we refer to [2, 38]. Convergence rates, without any constraints on the grid ratios, of many conservative finite difference (CONFD) methods for the one-dimensional NLS equation were established in [15, 16, 24, 45]. In fact, both the conservative laws of the numerical methods and the one-dimensional Sobolev inequality in discrete version play an important role in obtaining the a priori estimates of the numerical solutions in the maximum norm. Then, based on the a priori estimates of the numerical solutions, the convergence results of the CONFD schemes are obtained by using the standard energy