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Index Iteration Theory for Brake Orbit Type Solutions and Applications

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. In this paper, we give a survey on the index iteration theory of an index theory for brake orbit type solutions and its applications in the study of brake orbit problems including the Seifert conjecture and the minimal period solution problems in brake orbit cases.

Key Words: Hamiltonian system, Index theory, Iteration theory, Lagrangian boundary problem, Seifert conjecture, Brake orbits.

AMS Subject Classifications: 58E05, 70H05, 34C25, 37J45

1 Introduction

Since 1990, the iteration theory of the Maslov-type index theory for symplectic paths has been systematically developed by Long and his research group [54–56]. It has become a powerful tool in the study of various problems on periodic solutions (or orbits) of non-linear Hamiltonian systems including: existence, multiplicity, and stability of periodic solution orbits [57, 59, 67, 68] and closed geodesic [5, 13, 14], stability problems of periodic orbits of n-body problems [35–37], Rabinowitz's minimal periodic solution conjecture [12,50–52], Conley's conjecture on sub-harmonic periodic orbits for second as well as first order Hamiltonian systems [27,32]. Recently, Long, Duan, and Zhu published a survey paper [15] on this topic. Interested readers are referred to this paper and references therein.

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Since the paper of [57] of Long, Zhang, and Zhu in 2006, motivated by the study of the existence, multiplicity and stability of brake orbit type periodic solutions of Hamiltonian systems, in order to get more precise information, the index theory and its iteration theory of the Maslov-type index for symplectic paths under brake orbit boundary conditions have been systematically developed. This index and its iteration theory can be used to study brake orbits problems of reversible Hamiltonian systems [47,48,57,73,74] as well as other related problems including Conley conjecture and minimal periodic solution problems in brake orbit case [42,72].

In this survey, we shall give an introduction to this Maslov-type index theory and its relationship with the Maslov index theory in Section 1. In Section 2, we shall describe the main results in the iteration theory of such an index theory. For applications, in Section 3, we shall introduce recent developments on brake orbit problem for compact convex reversible hypersurfaces in $\mathbf{R}^{2n}$, which yields some partial answer to the Seifert conjecture on the multiplicity of brake orbits proposed by Seifert in 1948 [64]. In Section 4, we shall briefly summarize the study of the minimal period solution problems of reversible Hamiltonian systems in brake orbit case.

In this paper, let $\mathbf{N}$, $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{C}$ denote the sets of natural integers, integers, rational numbers, real numbers and complex numbers respectively. Let $\mathbf{U}$ be the unit circle of the complex plane $\mathbf{C}$, i.e., $\mathbf{U}=\{z\in\mathbf{C}|\,|z|=1\}$.

2 A review on the Maslov-type index theory i_L for symplectic paths under Lagrangian boundary condition

2.1 The i_ω index theory for symplectic paths

We firstly give the definition of i_ω index for symplectic paths which was first introduced by Long in [54] of 1999, all the materials here with historical notes can be found in [56] of 2002 and the recent survey paper [15]. Let $(\mathbf{R}^{2n},\omega_0)$ be the standard symplectic vector space with coordinates $(x_1,\cdots,x_n,y_1,\cdots,y_n)$ and the standard symplectic form

$$\omega_0=\sum_{i=1}^n dx_i\wedge dy_i.$$

Let

$$J=\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

be the standard symplectic matrix, where I_n is the identity matrix in $\mathbf{R}^n$. The real symplectic group $\mathrm{Sp}(2n)$ is defined by

$$\mathrm{Sp}(2n)=\{M\in \mathrm{GL}(2n,\mathbf{R})|M^TJM=J\},$$