Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 2, Number 4, December 2020, 485–493 DOI:10.12150/jnma.2020.485

Eigenvalues of Fourth-order Singular Sturm-Liouville Boundary Value Problems*

Lina Zhou[1,†], Weihua Jiang[2] and Qiaoluan Li[1]

Abstract In this paper, by using Krasnoselskii's fixed-point theorem, some sufficient conditions of existence of positive solutions for the following fourth-order nonlinear Sturm-Liouville eigenvalue problem:

$$\begin{cases} \frac{1}{p(t)}(p(t)u''')'(t) + \lambda f(t,u) = 0, t \in (0,1), \\ u(0) = u(1) = 0, \\ \alpha u''(0) - \beta \lim_{t\to 0^+} p(t)u'''(t) = 0, \\ \gamma u''(1) + \delta \lim_{t\to 1^-} p(t)u'''(t) = 0, \end{cases}$$

are established, where $\alpha, \beta, \gamma, \delta \geq 0$, and $\beta\gamma + \alpha\gamma + \alpha\delta > 0$. The function p may be singular at $t = 0$ or 1, and f satisfies Carathéodory condition.

Keywords Sturm-Liouville problems, Eigenvalue, Krasnoselskii's fixed-point theorem.

MSC(2010) 34B15, 34B25.

1. Introduction

In this paper, we will study the existence of positive solutions for the following fourth-order nonlinear Sturm-Liouville eigenvalue problem:

$$\begin{cases} \frac{1}{p(t)}(p(t)u''')'(t) + \lambda f(t,u) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \\ \alpha u''(0) - \beta \lim_{t\to 0^+} p(t)u'''(t) = 0, \\ \gamma u''(1) + \delta \lim_{t\to 1^-} p(t)u'''(t) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\alpha, \beta, \gamma, \delta \geq 0$ are some constants satisfying $\beta\gamma + \alpha\gamma + \alpha\delta > 0$, $p \in C^1((0,1),(0,+\infty))$ satisfying $\int_0^1 \frac{ds}{p(s)} < +\infty$, and $f : [0,1] \times R^+ \to R^+$ satisfies Carathéodory condition. From the above conditions, the function p may be singular at $t = 0$ or 1.

[†]the corresponding author.
Email address: lnazhou@163.com(L. Zhou)
[1]School of Mathematical Science, Hebei Normal University, Shijiazhuang, Hebei 050024, China
[2]College of Science, Hebei University of Science and Technology, Shijiazhuang, Hebei 050018, China
[*]The authors were supported by the Natural Science Foundation of China (11971145, 11775169), the Natural Science Foundation of Hebei Province (A2019205133, A2018208171).

Sturm-Liouville boundary problems have been widely investigated in various fields, such as mathematics, physics and meteorology. In recent decades, a vast amount of research was done on the existence of positive solutions of Sturm-Liouville boundary value problems. Within this development, they paid attention to the theory of eigenvalues and eigenfunctions of Sturm-Liouville problems [2-18]. In particular, many authors were interested in the nonlinear singular Sturm-Liouville problems [10-16]. In [10], Yao et al. proved that the BVP (1.1) has one or two positive solutions for some λ under the assumptions $f_0 = f_\infty = 0$ or $f_0 = f_\infty = \infty$. In [13], by a new comparison theorem, Zhang et al. proved that the BVP(1.1) has at least a positive solution for large enough λ under the assumptions:
(1) $p \in C^1((0,1),(0,+\infty))$ and $\int_0^1 \frac{ds}{p(s)} < +\infty$;
(2) $f(t,u) \in C((0,1)\times(0,+\infty),[0,+\infty))$ is decreasing in u ;
(3) For any $\mu > 0$, $f(t,\mu) \neq 0$ and $0 < \int_0^1 k(s)p(s)f(s,\mu s(1-s))ds < +\infty$;
(4) For any $u \in [0,+\infty)$, $\lim\limits_{\mu\to+\infty} \mu f(t,\mu u) = +\infty$ uniformly on $t \in (0,1)$.

In this paper, we consider the existence of positive solutions of the BVP(1.1), under the following conditions:
(H_1) $p \in C^1((0,1),(0,+\infty))$ and $\int_0^1 \frac{ds}{p(s)} < +\infty$;
(H_2) $f : [0,1]\times R^+ \to R^+$ satisfies Carathéodory condition, that is $f(\cdot,u)$ is measurable for each fixed $u \in R^+$, and $f(t,\cdot)$ is continuous for a.e. $t \in [0,1]$;
(H_3) for any $r > 0$, there exists $h_r(t) \in L^1[0,1]$, such that $f(t,u) \leq h_r(t)$, a.e. $t \in [0,1]$, where $u \in [0,r]$, and $0 < \int_0^1 k(s)p(s)h_r(s) < +\infty$.

By Krasnoselskii's fixed-point theorem, two main results are obtained under $(H_1) - (H_3)$.

2. Preliminaries

In this section, we present some necessary definitions, theorems and lemmas.

Definition 2.1. A function u is called a solution of the BVP(1.1) if $u \in C^3([0,1],[0,+\infty))$ satisfies $p(t)u'''(t) \in C^1([0,1],[0,+\infty))$ and the BVP(1.1). Also, u is called a positive solution if $u(t) > 0$ for $t \in [0,1]$ and u is a solution of the BVP (1.1). For some λ, if the BVP (1.1) has a positive solution u, then λ is called an eigenvalue and u is called a corresponding eigenfunction of the BVP (1.1).

Theorem 2.1. ([1], [19]) *Let X be a real normal linear space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are relatively open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : \overline{\Omega}_2 \to P$ be a completely continuous operator such that, either*
(1) $\| Tu \| \leq r_1, u \in \partial\Omega_1$; $\| Tu \| \geq r_2, u \in \partial\Omega_2$ *or*
(2) $\| Tu \| \geq r_1, u \in \partial\Omega_1$; $\| Tu \| \leq r_2, u \in \partial\Omega_2$.
Then T has a fixed point in $P \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$.

In this paper, we always make the following assumption:
(H_1) $p \in C^1((0,1),(0,+\infty))$ and $\int_0^1 \frac{ds}{p(s)} < +\infty$.

Now we denote by $H(t,s)$ and $G(t,s)$, respectively, the Green's functions for the following boundary value problems:

$$\begin{cases} -u'' = 0, 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$