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Shehu Transform: Extension to Distributions and Measures

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Abstract This paper improves the computational aspect of the Shehu transform. An inversion formula is given. Finally the Shehu transform is extended to distributions and measures.

Keywords Shehu transform, Laplace transform, Integral transform, Distribution, Measure.

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1. Introduction

Integral transforms have shown their usefulness in mathematics and engineering. They are effective and ubiquitous in many areas such as harmonic analysis, signal processing, differential equations, etc. The family of the integral transforms is very wide but the most famous are the Fourier transform and the Laplace transform. However, in order to solve some recent problems many authors introduced some Fourier-like and Laplace-like transforms (see for instance [9, 10, 12, 13]). In [12], authors introduced a new integral transform which generalizes the Laplace transform [6] and the Yang transform [13]. They called it *Shehu transform*. Very soon, this transformation became a powerful tool in applied science, more precisely in the field of differential equations.

The Shehu transform is defined by

$$\mathbb{S}[f(t)](s,u) = \int_0^\infty e^{\frac{-st}{u}} f(t)dt,\ s>0,\ u>0. \tag{1.1}$$

while the Laplace transform is given by

$$\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)dt \tag{1.2}$$

and the Yang transform is defined by

$$\mathcal{Y}[f(t)](u) = \int_0^\infty e^{\frac{-t}{u}} f(t)dt. \tag{1.3}$$

Evidently, when $u=1$ the Shehu transform becomes the Laplace transform (real variables) and for $s=1$ the Shehu transform is the Yang transform. In [12] the

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authors used the Shehu transform to solve different types of ordinary and partial differential equations. Many properties of the Shehu transform and its applications to engineering problems have been investigated by Aggarwal et al. [1–3]. Some recent publications concerning the Shehu transform are [4, 5, 7, 8, 11]. For instance, in [8] the authors present new properties of this transform. They apply this transformation to Atangana-Baleanu derivatives in Caputo and in Riemann-Liouville senses to solve some fractional differential equations and authors in [4] proposed a reliable and new algorithm for solving time-fractional differential models arising from physics and engineering.

Our first intention here is to put the Shehu transform at the heart of functional analysis and to give it a status close to that of the Fourier transform or the Laplace transform. The main aim of this paper is to improve the computational aspect of the Shehu transform and to extend it to distributions and measures.

The rest of the paper is organized as follows. In Section 2, we relate the Shehu transform to the Laplace transform and indicate how to improve the computation of the Shehu transform of functions using basic calculus and we give an inversion formula. In Section 3, we discuss some sufficient conditions for the existence of the Shehu transform of a function/signal. In Section 4, we extend the Shehu transform to distributions and finally in Section 5 we extend it to measures.

2. Computation and inversion formula

The following relation connects the Shehu transform to the Laplace transform.

$$\mathbb{S}[f(t)](s,u) = \mathcal{L}[f(t)]\left(\frac{s}{u}\right). \tag{2.1}$$

Therefore, we deduce the following consequences:

1. The main properties of the Shehu transform can be obtained easily from the Laplace transform.
2. One can compute the Shehu transform of a function from its Laplace transform quickly. Then, we were able complete the table of the transform of usual functions in [12].

Let us give some examples.

Example 2.1.

1. Choose $f(t) = \cos(\alpha t)$. We know that $\mathcal{L}[\cos(\alpha t)](s) = \dfrac{s}{s^2+\alpha^2}$.
   Then $\mathbb{S}[\cos(\alpha t)](s,u) = \dfrac{\frac{s}{u}}{(\frac{s}{u})^2+\alpha^2} = \dfrac{su}{s^2+\alpha^2u^2}$.
2. Choose $f(t) = \dfrac{t^n}{n!}$. We know that $\mathcal{L}\left[\dfrac{t^n}{n!}\right](s) = \dfrac{1}{s^{n+1}}$.
   Then $\mathbb{S}\left[\dfrac{t^n}{n!}\right](s,u) = \dfrac{1}{(\frac{s}{u})^{n+1}} = \left(\dfrac{u}{s}\right)^{n+1}$.
3. Choose $f(t) = J_0(\alpha t)$ (Bessel function). One knows that $\mathcal{L}\left[J_0(\alpha t)\right](s) = \dfrac{1}{\sqrt{s^2+\alpha^2}}$. Then $\mathbb{S}[J_0(\alpha t)](s,u) = \dfrac{1}{\sqrt{(\frac{s}{u})^2+\alpha^2}} = \dfrac{u}{\sqrt{s^2+\alpha^2u^2}}$.

At the end of this article, there is a table containing the Shehu transforms of many functions computed in the above way.