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Global Attractor of Hindmarsh-Rose Equations in Neurodynamics*

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Abstract Global dynamics for a new mathematical model in neurodynamics of the diffusive Hindmarsh-Rose equations on a bounded domain is investigated in this paper. The existence of a global attractor and its regularity are proved through uniform estimates showing the dissipative properties and the asymptotically compact and smoothing characteristics.

Keywords Diffusive Hindmarsh-Rose equations, Global attractor, Absorbing property, Asymptotic compactness, Attractor regularity.

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1. Introduction

The Hindmarsh-Rose equations as a three-dimensional mathematical model for neuronal spiking-bursting of the intracellular membrane potential observed in experiments was originally proposed in [11]. This model composed of three coupled ordinary differential equations has been studied through numerical simulations and bifurcation analysis in recent years, cf. [13, 15, 26, 32] and the references therein. It exhibits rich and interesting bursting patterns, especially chaotic bursting and dynamics such as self-excitation and self-oscillations.

In this work we present and study the global dynamics of the diffusive Hindmarsh-Rose equations as a new PDE model in neurodynamics:

$$\frac{\partial u}{\partial t} = d_1 \Delta u + \varphi(u) + v - w + J, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + \psi(u) - v, \tag{1.2}$$

$$\frac{\partial w}{\partial t} = d_3 \Delta w + q(u - c) - rw, \tag{1.3}$$

for $t > 0,\ x \in \Omega \subset \mathbb{R}^n$ $(n \le 3)$, where Ω is a bounded domain with locally Lipschitz continuous boundary. The nonlinear terms

$$\varphi(u) = au^2 - bu^3, \quad \text{and} \quad \psi(u) = \alpha - \beta u^2. \tag{1.4}$$

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In this system, the variable $u(t,x)$ refers to the membrane electric potential of a neuron cell, the variable $v(t,x)$ represents the transport rate of the ions of sodium and potassium through the fast channels and is called the spiking variable, while the variables $w(t,x)$ represents the transport rate across the cell membrane through slow channels of calcium and other ions correlated to the bursting phenomena and is called the bursting variable. The inject current J is treated as a constant.

All the involved parameters are positive constants except $c \in \mathbb{R}$ in the w-equation, which is a reference value of the membrane potential of neuron cells. In the original model [32], a set of typical parameters are

$$J = 3.281, \quad r = 0.0021, \quad S = 4.0, \quad q = rS, \quad c = -1.6,$$
$$\varphi(s) = 3.0\, s^2 - s^3, \quad \psi(s) = 1.0 - 5.0\, s^2.$$

We impose the Neumann boundary conditions for the three components,

$$\frac{\partial u}{\partial \nu}(t,x) = 0, \quad \frac{\partial v}{\partial \nu}(t,x) = 0, \quad \frac{\partial w}{\partial \nu}(t,x) = 0, \qquad t > 0,\ x \in \partial\Omega, \tag{1.5}$$

and the initial conditions to be specified later,

$$u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x), \qquad x \in \Omega. \tag{1.6}$$

1.1. The Hindmarsh-Rose model in ODE

The original Hindmarsh-Rose model was developed [11] in 1984,

$$\begin{aligned}
\frac{du}{dt} &= au^2 - bu^3 + v - w + J,\\
\frac{dv}{dt} &= \alpha - \beta u^2 - v,\\
\frac{dw}{dt} &= q(u - u_R) - rw.
\end{aligned} \tag{1.7}$$

and was motivated by the discovery of neuronal cells in the pond snail *Lymnaea* which generated a burst after being depolarized by a short current pulse. This model characterizes the phenomena of synaptic bursting and especially chaotic bursting in a three-dimensional (u,v,w) space, which incorporates a third variable representing a slow current that hyperpolarizes the neuronal cell. This neurodynamics model is different from the four-dimensional highly nonlinear Hodgkin-Huxley equations [12] (1952) and from the two-dimensional FitzHugh-Nagumo equations [10] (1961-1962) for neuron dynamics in self-excitation and oscillation. The 2D FitzHugh-Nagumo model admits exquisite phase plane analysis showing sustained periodic spiking with refractory period, but it excludes chaotic solutions so that no chaotic bursting can be generated.

Neuronal signals are electrical pulses called spikes or the action potential. Neuron bursting of alternating phases of rapid firing spikes and then quiescence constitutes a mechanism to modulate and pace-setting for brain functionalities and to communicate signals with the neighbor or remote neurons. Bursting patterns occur in a variety of bio-systems such as pituitary melanotropic gland, thalamic neurons, respiratory pacemaker neurons, and insulin-secreting pancreatic β-cells, cf. [2, 3, 5, 11].