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Stability Analysis for the Numerical Simulation of Hybrid Stochastic Differential Equations*

Guangjie Li[1,2] and Qigui Yang[2,†]

Abstract This paper is mainly concerned with the exponential stability of a class of hybrid stochastic differential equations–stochastic differential equations with Markovian switching (SDEwMSs). It first devotes to reveal that under the global Lipschitz condition, a SDEwMS is pth ($p \in (0, 1)$) moment exponentially stable if and only if its corresponding improved Euler-Maruyama(IEM) method is pth moment exponentially stable for a suitable step size. It then shows that the SDEwMS is pth($p \in (0, 1)$) moment exponentially stable or its corresponding IEM method with small enough step sizes implies the equation is almost surely exponentially stable or the corresponding IEM method, respectively. In that sense, one can infer that the SDEwMS is almost surely exponentially stable and the IEM method, no matter whether the SDEwMS is pth moment exponentially stable or the IEM method. An example is demonstrated to illustrate the obtained results.

Keywords Moment exponential stability, almost sure exponential stability, Markovian switching, improved Euler-Maruyama method.

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1. Introduction

Stochastic differential equations (SDEs) have been widely used in many branches of science and industry. Stability analysis as a particular interest of SDEs has aroused special attention of many scholars (see [2,8,16,17] and the literature cited therein). In the study of stochastic stability, Lyapunov functions technique is the classical and powerful technique. However, in general, it is not convenient for us to use this method for there is no an universal method can guarantee to find an appropriate Lyapunov function, which motivates us to employ numerical methods with sufficiently small step sizes to study the stochastic stability. Hence, for SDEs, many investigators have paid a deal of attention to stability analysis of numerical methods (e.g. [5,7,13,14,19]). Moreover, the following two questions are concerning:

(Q1) If a SDE is stochastically stable, will the numerical method be stochastically stable?

† the corresponding author.
Email address:scutliguangjie@163.com(G. Li), qgyang@scut.edu.cn(Q. Yang)
1 Department of Mathematics, Shenzhen Technology University, Shenzhen, Guangdong 518118, China
2 School of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, China
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(Q2) Conversely, if the numerical method is stochastically stable, will the SDE is stochastically stable?

Two natural and important notions of stochastical stability are moment exponentially stable and almost surely exponentially stable. Moment exponential stability, also known as the pth moment exponential stability. In the case of $p = 2$, moment exponential stability means exponential stability in mean square. When the stochastic stability is understood as the mean square exponential stability, answers to $(Q1)$ and $(Q2)$ can be found in [3, 13]. Higham et al. [4] showed that under the global Lipscitz condition, mean square exponential stability of SDEs and that of the numerical method with sufficiently small step sizes are equivalent, implying that answers to $(Q1)$ and $(Q2)$ are positive. For the stochastic stability means the almost sure exponential stability, there are a lot of results for $(Q1)$, but few answers to $(Q2)$. Higham et al. [5] is the first paper that discussed both $(Q1)$ and $(Q2)$ for a reasoned class of SDEs. For the linear scalar SDEs, they presented positive answers to $(Q1)$ and $(Q2)$ by the Euler-Maruyama (EM) method. For the nonlinear SDEs with the linear growth condition and some additional conditions, they also answered $(Q1)$ using the EM method. For the nonlinear SDEs without the linear growth condition, but required the drift coefficient obeys the one-sided Lipschitz condition, the answer to $(Q1)$ alone is positive through the backward Euler-Maruyama (BEM) method. Recently, Mao [9] proved that under the global Lipschitz condition, the almost sure exponential stability of the SDEs is shared with that the stochastic theta method, and therefore presented positive answers to $(Q1)$ and $(Q2)$.

As is known, hybrid stochastic differential equations have increasingly gained attention in biological systems, financial engineering, wireless communications and so forth (see [18, 20]). One of the important classes of the hybrid stochastic differential equations is the SDEwMSs. Generally, most of SDEwMSs can rarely be solved explicitly and hence numerical approximation becomes an important tool in studying them. When the SDEwMS is considered in the $(Q1)$ and $(Q2)$, Pang et al. [12] proved that under appropriate conditions, the EM method with sufficiently small step sizes can capture the almost sure and the pth moment exponential stability of the linear scalar SDEwMS, they therefore gave the positive answer to $(Q1)$, but not to $(Q2)$. For the nonlinear SDEwMSs, the authors in [10] showed that without the global Lipschitz condition, the BEM method may capture the almost sure exponential stability, they also positively answered $(Q1)$ but did not address $(Q2)$.

Although a lot of results on addressing $(Q1)$ and $(Q2)$ for SDEs have been obtained (see [6, 19] and the references therein), unfortunately, there are almost no answers to $(Q1)$ and $(Q2)$ for SDEwMSs due to the difficulty in dealing with the Markovian switching. Therefore, it is significant to investigate $(Q1)$ and $(Q2)$ for SDEwMSs. Motivated by Mao [9], this paper first shows that under the global Lipschitz condition, the SDEwMS is pth $(p \in (0, 1))$ moment exponentially stable if and only if the IEM method with a sufficiently small step size is pth moment exponentially stable. Based on such result, we can positively answer both $(Q1)$ and $(Q2)$ for the SDEwMS when the stochastic stability means exponential stability in the sense of pth $(p \in (0, 1))$ moment. This paper then proves that the SDEwMS is pth$(p \in (0, 1))$ moment exponentially stable or the IEM method implies the SDEwMS is almost surely exponentially stable or the IEM method, respectively. Moreover, the obtained theory ensures that either the SDEwMS is pth moment exponentially stable or the IEM method, one can assert that the almost sure expo-