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Near-invariant Tori on Exponentially Long Time for Poisson systems*

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Abstract This paper deals with the near-invariant tori for Poisson systems. It is shown that the orbits with the initial points near the Diophantine torus approach some quasi-periodic orbits over an extremely long time. In particular, the results hold for the classical Hamiltonian system, and in this case the drift of the motions is smaller than one in the past works.

Keywords Poisson system, near-invariant tori, rapidly Newton iteration.

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1. Problem, preliminaries and result

The problem of stability of Hamiltonian systems occupies a crucial place in the field of dynamic systems. As well known, KAM theory shows that most of quasi-periodic motions of the integrable Hamiltonian systems are persistent under a small perturbation. The name comes from the initials of Kolmogorov, Arnold and Moser who laid the foundation of the theory [1, 3, 6]. In 1977s, Nekhoroshev presented a global result. He showed that under a perturbation of order ε of an integrable Hamiltonian system with the steepness condition, the action variable of an arbitrary orbit vary only in the order of ε^b over a time interval of the order of $\exp(\varepsilon^{-a})$, where a and b are positive constants [7]. Now one refers to Nekhoroshev's theorem as effective stability. Later on, much mathematics are devoted to studying KAM theory and effective stability, and a great deal of significant results are obtained, see [2, 4, 8–10] and the references therein.

One remarkable problem is that the above works only localize on classical Hamiltonian systems which are defined on an even-dimensional manifold. Many systems in applications can not be written as Hamiltonian forms, for example, Lotka-Voterra model [11], the motion equation of a rigid body without any external forces, ABC flow and so on. The reason is that their phase spaces are of odd-dimensional. Note

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that these systems possess general Poisson structures. The problem considered in this paper is to generalize the stability theory of Hamiltonian systems to Poisson systems defined on odd-dimensional spaces.

We first introduce the concept of Poisson systems. Moreover, some fundamental properties are given without proofs. For details, see [5].

Let $B : D \times T^n \to R^{(m+n)\times(m+n)}$ be a smooth matrix-valued function, where $D \subset R^m$ be a bounded, connected and closed region, and $T^n = R^n/Z^n$. For all $z = (y, x) \in D \times T^n$, set

$$\{F, G\}(z) = \nabla F(z)^T B(z)\nabla G(z). \tag{1.1}$$

Lemma 1.1. *The bracket defined in (1.1) is bilinear, skew-symmetric and satisfies*

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0, \tag{1.2}$$

$$\{F \cdot G, H\} = F \cdot \{G, H\} + G \cdot \{F, H\} \tag{1.3}$$

if and only if $B^T = -B$ *and for all* i, j, k,

$$\sum_{l=1}^{m+n} \left(\frac{\partial b_{ij}(z)}{\partial z_l} b_{lk}(z) + \frac{\partial b_{jk}(z)}{\partial z_l} b_{li}(z) + \frac{\partial b_{ki}(z)}{\partial z_l} b_{lj}(z) \right) = 0. \tag{1.4}$$

Definition 1.1. If $B(z)$ satisfies $B^T = -B$ and (1.4), formula (1.1) is said to represent a general Poisson bracket. The corresponding system

$$\dot{z} = B(z)\nabla H(z) \tag{1.5}$$

is said to be a Poisson system with Hamiltonian H.

Definition 1.2. A transformation $\varphi : U \to R^{m+n}$ (where U is an open set in R^{m+n}) is called a Poisson change with respect to the bracket (1.1), if the structure matrix B satisfies

$$\varphi'(z)B(z)\varphi'(z)^T = B(\varphi(z)).$$

Lemma 1.2. *If* $B(z)$ *is the structure matrix of a Poisson bracket, the flow* $\phi^t(z)$ *of (1.5) is a Poisson change.*

Lemma 1.3. *Let* $\phi^t(z)$ *be a flow of (1.5). Acting on a function* $F : R^{m+n} \to R$, *the following formula holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\phi^t(z)) = \{F, G\}(\phi^t(z)).$$

Definition 1.3. Let F and G be two smooth functions defined on some open subset of R^{m+n}. F and G are said to be in involution, if $\{F, G\} = 0$.

From now on, we begin to describe the main result of this paper. Consider a Poisson system

$$\dot{z} = B(y)\nabla H(z) \tag{1.6}$$

defined on some complex neighborhood of $D\times T^n$ in $C^m \times C^n$, where B is a structure matrix independent of x.

Through this paper, we assume that y_j, $j = 1, \cdots, m$, and x_k, $k = 1, \cdots, n$, respectively, satisfy the involution condition:

$$\{y_i, y_j\} = 0, \quad i, j = 1, \cdots, m, \tag{1.7}$$